

### CALCULUS OF VARIATION ASSIGNMENT

DECEMBER – 2015

#### DECEMBER – 2014

##### PART - B

1. Consider the functional

$$I(y) = y^2(1) + \int_0^1 y'^2(x)dx, \quad y(0) = 1,$$

where  $y \in C^2([0,1])$ . If  $y$  extremizes  $J$ , then

- |                                |                                |
|--------------------------------|--------------------------------|
| 1. $y(x) = 1 - \frac{1}{2}x^2$ | 2. $y(x) = 1 - \frac{1}{2}x$   |
| 3. $y(x) = 1 + \frac{1}{2}x$   | 4. $y(x) = 1 + \frac{1}{2}x^2$ |

##### PART - C

2. Let  $y \in C^2([0, \pi])$  satisfying

$$y(0) = y(\pi) = 0 \text{ and } \int_0^\pi y^2(x)dx = 1$$

extremize the functional

$$J(y) = \int_0^\pi (y'(x))^2 dx; \quad y' = \frac{dy}{dx}. \text{ Then}$$

- |   |  |
|---|--|
| 1. $y(x) = \sqrt{\frac{2}{\pi}} \sin x$ | 2. $y(x) = -\sqrt{\frac{2}{\pi}} \sin x$ |
| 3. $y(x) = \sqrt{\frac{2}{\pi}} \cos x$ | 4. $y(x) = -\sqrt{\frac{2}{\pi}} \cos x$ |

#### JUNE – 2015

##### PART - C

3. The extremal of the functional

$$\int_0^\alpha (y'^2 - y^2)dx \text{ that passes through } (0,0) \text{ and}$$

$(\alpha,0)$  has a

1. weak minimum if  $\alpha < \pi$ .
2. strong minimum if  $\alpha < \pi$ .
3. weak minimum if  $\alpha > \pi$ .
4. strong minimum if  $\alpha > \pi$ .

4. The extremal of the functional  $I = \int_0^{x_1} y^2(y')^2 dx$

that passes through  $(0,0)$  and  $(x_1, y_1)$  is

1. a constant function
2. a linear function of  $x$
3. part of parabola
4. part of an ellipse

##### PART - B

5. The functional  $I(y(x)) = \int_a^b (y^2 + y'^2 - 2y \sin x) dx$ , has the following extremal with  $C_1$  and  $C_2$  as arbitrary constants.

1.  $y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{2} \sin x$ .
2.  $y = C_1 e^x + C_2 e^{-x} + \frac{1}{2} \sin x$ .
3.  $y = C_1 e^x + C_2 e^{-x} - \frac{1}{2} \sin x$ .
4.  $y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{2} \cos x$ .

##### PART - C

6. To show the existence of a minimizer for the functional  $J[y] = \int_a^b f(x, y, y') dx$ , for which there is a minimizing sequence  $(\varphi_n)$ , it is enough to have

1.  $(\varphi_n)$  is convergent and  $J$  is continuous.
2.  $(\varphi_n)$  is convergent and  $J$  is differentiable.
3.  $(\varphi_n)$  has a convergent subsequence and  $J$  is continuous.
4.  $(\varphi_n)$  has a convergent subsequence and  $J$  is differentiable.

#### JUNE – 2016

##### PART - B

7. The curve of fixed length  $l$ , that joins the points  $(0,0)$  and  $(1,0)$ , lies above the  $x$ -axis, and encloses the maximum area between itself and the  $x$ -axis, is a segment of

1. a straight line.
2. a parabola
3. an ellipse
4. a circle

##### PART - C

8. Let  $y = y(x)$  be the extremal of the functional

$$I[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \text{ subject to the}$$

condition that the left end of the extremal moves along  $y = x^2$ , while the right end moves along  $x - y = 5$ , Then the

1. shortest distance between the parabola and the straight line is  $\left(\frac{19\sqrt{2}}{8}\right)$ .
2. slope of the extremal at  $(x,y)$  is  $\left(-\frac{3}{2}\right)$ .
3. point  $\left(\frac{3}{4}, 0\right)$  lies on the extremal.
4. extremal is orthogonal to the curve  $y = \frac{x}{2}$ .

### DECEMBER – 2016

#### PART - B

9. If  $J[y] = \int_1^2 (y'^2 + 2yy' + y^2) dx$ ,  $y(1) = 1$  and  $y(2)$  is arbitrary then the extremal is
1.  $e^{x-1}$
  2.  $e^{x+1}$
  3.  $e^{1-x}$
  4.  $e^{-x-1}$

#### PART - C

10. The functional  $J[y] = \int_0^1 (y'^2 + x^2) dx$  where  $y(0) = -1$  and  $y(1) = 1$  on  $y = 2x - 1$ , has
1. weak minimum
  2. weak maximum
  3. strong minimum
  4. strong maximum
11. Let  $y(x)$  be a piecewise continuously differentiable function on  $[0,4]$ . Then the functional  $J[y] = \int_0^4 (y' - 1)^2 (y + 1)^2 dx$  attains minimum if  $y = y(x)$  is
1.  $y = \frac{x}{2} \quad 0 \leq x \leq 4$
  2.  $y = \begin{cases} -x & 0 \leq x \leq 1 \\ x - 2 & 1 \leq x \leq 4 \end{cases}$
  3.  $y = \begin{cases} 2x & 0 \leq x \leq 2 \\ -x + 6 & 2 \leq x \leq 4 \end{cases}$
  4.  $y = \begin{cases} x & 0 \leq x \leq 3 \\ -x + 6 & 3 \leq x \leq 4 \end{cases}$

### JUNE – 2017

#### PART - B

12. The infimum of  $\int_0^1 (u'(t))^2 dt$  on the class of functions  $\{u \in C^1[0,1] \text{ such that } u(0) = 0 \text{ and } \max_{[0,1]} |u| = 1\}$  is equal to
1. 0
  2. 1/2
  3. 1
  4. 2
13. Consider the functional  $I(y(x)) = \int_{x_0}^{x_1} f(x, y) \sqrt{1 + y'^2} e^{\tan^{-1} y'} dx$  where  $f(x, y) \neq 0$ . Let the left end of the extremal be fixed at the point  $A(x_0, y_0)$  and the right end  $B(x_1, y_1)$  be movable along the curve  $y = \psi(x)$ . Then the extremal  $y = y(x)$  intersects the curve  $y = \psi(x)$  along which the boundary point  $B(x_1, y_1)$  slides at an angle
1.  $\pi/3$
  2.  $\pi/2$
  3.  $\pi/4$
  4.  $\pi/6$

### DECEMBER – 2017

#### PART - C

14. Let  $X = \{u \in C^1[0,1] \mid u(0) = 0\}$  and let  $I: X \rightarrow \mathbb{R}$  be defined as  $I(u) = \int_0^1 (u'(t))^2 - u(t)^2 dt$ . Which of the following are correct?
1.  $I$  is bounded below
  2.  $I$  is not bounded below
  3.  $I$  attains its infimum
  4.  $I$  does not attain its infimum
15. Let  $I: C^1[0,1] \rightarrow \mathbb{R}$  be defined as  $I(u) = \frac{1}{2} \int_0^1 (u'(t))^2 - 4\pi^2 u(t)^2 dt$ . Let us set  $(P)m = \inf \{I(u) : u \in C^1[0,1] : u(0) = u(1) = 0\}$ . Let  $\bar{u} \in C^1[0,1]$  satisfy the Euler – Lagrange Equation associated with (P). Then
1.  $m = -\infty$  i.e.,  $I$  is not bounded below
  2.  $m \in \mathbb{R}$ , with  $I(\bar{u}) = m$
  3.  $m \in \mathbb{R}$ , with  $I(\bar{u}) > m$
  4.  $m \in \mathbb{R}$ , with  $I(\bar{u}) < m$

### JUNE - 2018

#### PART - B

16. Consider  $J[y] = \int_0^1 [(y')^2 + 2y] dx$  subject to  $y(0) = 0, y(1) = 1$ . Then  $\inf J[y]$
- |                       |                       |
|-----------------------|-----------------------|
| 1. is $\frac{23}{12}$ | 2. is $\frac{21}{24}$ |
| 3. is $\frac{18}{25}$ | 4. does not exist     |

#### PART - C

17. The extremal of the functional  $J[y] = \int_0^1 y'^2(x) dx$  subject to  $y(0)=0, y(1)=1$  and  $\int_0^1 y(x) dx = 0$  is
- |   |
|---|
| 1. $3x^2 - 2x$                                      |
| 2. $8x^3 - 9x^2 + 2x$                               |
| 3. $\frac{5}{3}x^4 - \frac{2}{3}x$                  |
| 4. $\frac{-21}{2}x^5 + 10x^4 + 4x^3 - \frac{5}{2}x$ |
18. The admissible extremal for  $J[y] = \int_0^{\log 3} [e^{-x} y'^2 + 2e^x (y' + y)] dx$ , where  $y(\log 3) = 1$  and  $y(0)$  is free is
- |              |                  |
|--------------|------------------|
| 1. $4 - e^x$ | 2. $10 - e^{2x}$ |
| 3. $e^x - 2$ | 4. $e^{2x} - 8$  |

### DECEMBER - 2018

#### PART-B

19. Consider the functional  $J[y] = \int_0^2 (1 - y'^2)^2 dx$  defined on  $\{y \in C[0,2]: y$  is piecewise  $C^1$  and  $y(0)=y(2) = 0\}$ . Let  $y_e$  be a minimize of the above functional. Then  $y_e$  has
- a unique corner point
  - two corner points
  - more than two corner points
  - no corner points

#### PART-C

20. Consider the functional  $J[y] = \int_0^1 [(y')^2 - (y')^4] dx$  subject to  $y(0)=0, y(1) = 0$ . A broken extremal is a continuous extremal whose derivative has jump discontinuities at a finite number of

points. Then which of the following statements are true?

- There are no broken extremals and  $y = 0$  is an extremal.
- There is a unique broken extremal.
- There exist more than one and finitely many broken extremals.
- There exist infinitely many broken extremals.

21. The extremals of the functional

$$J[y] = \int_0^1 [720x^2 y - (y'')^2] dx, \text{ subject to}$$

$$y(x) = y'(0) = y(1) = 0, y'(1) = 6, \text{ are}$$

- |                        |                        |
|------------------------|------------------------|
| 1. $x^6 + 2x^3 - 3x^2$ | 2. $x^5 + 4x^4 - 5x^3$ |
| 3. $x^5 + x^4 - 2x^3$  | 4. $x^6 + 4x^3 - 6x^2$ |

### JUNE - 19

#### PART - B

22. Let  $x^*(t)$  be the curve which minimizes the functional

$$J(x) = \int_0^1 [x^2(t) + \dot{x}^2(t)] dt$$

satisfying  $x(0)=0, x(1) = 1$ . Then the value

of  $x^*\left(\frac{1}{2}\right)$  is

- |                            |                             |
|----------------------------|-----------------------------|
| 1. $\frac{\sqrt{e}}{1+e}$  | 2. $\frac{2\sqrt{e}}{1+e}$  |
| 3. $\frac{\sqrt{e}}{1+2e}$ | 4. $\frac{2\sqrt{e}}{1+2e}$ |

#### PART - C

23. Let  $B = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$  and  $C = \{(x, y) | x^2 + y^2 = 1\}$  and let  $f$  and  $g$  be continuous functions. Let  $u$  be the minimizer of the functional

$$J[v] = \iint_B (v_x^2 + v_y^2 - 2fv) dx dy + \int_C (v^2 - 2gv) ds.$$

Then  $u$  is a solution of

- $-\Delta u = f, \frac{\partial u}{\partial n} + u = g$
- $\Delta u = f, \frac{\partial u}{\partial n} - u = g$
- $-\Delta u = f, \frac{\partial u}{\partial n} = g$
- $\Delta u = f, \frac{\partial u}{\partial n} = g$

where  $\frac{\partial u}{\partial n}$  denotes the directional derivative of  $u$  in the direction of the outward drawn normal at  $(x, y) \in C$

24. Consider the functional  $J[y] = \int_0^1 [(y'(x))^2 + (y'(x))^3] dx$ , subject to  $y(0) = 1$  and  $y(1) = 2$ . Then
1. there exists an extremal  $y \in C^1([0, 1]) \setminus C^2([0, 1])$
  2. there exists an extremal  $y \in C([0, 1]) \setminus C^1([0, 1])$
  3. every extremal  $y$  belongs to  $C^1([0, 1])$
  4. every extremal  $y$  belongs to  $C^2([0, 1])$

**DECEMBER – 2019**

**PART-B**

25. Let  $y = \phi(x)$  be the extremizing function for the functional
- $$I(y) = \int_0^1 y^2 \left( \frac{dy}{dx} \right)^2 dx, \text{ subject to}$$
- $y(0) = 0, y(1) = 1$ . Then  $\phi(1/4)$  is equal to
1. 1/2
  2. 1/4
  3. 1/8
  4. 1/12

**PART-C**

26. Let  $y = y(x) \in C^4([0, 1])$  be an extremizing function for the functional
- $$I(y) = \int_0^1 \left[ \left( \frac{d^2 y}{dx^2} \right)^2 - 2y \right] dx, \text{ satisfying}$$
- $y(0) = 0 = y(1)$ . Then an extremal  $y(x)$ , satisfying the given conditions at 0 and 1 together with the natural boundary conditions, is given by
1.  $\frac{x}{24}(x-1)^3$
  2.  $\frac{x^2}{24}(x-1)^2$
  3.  $\frac{x}{24}(x^3 - 2x^2 + 1)$
  4.  $\frac{x}{24}(x^3 + x^2 - 2)$

27. The minimum value of the functional

$$I(y) = \int_0^\pi \left( \frac{dy}{dx} \right)^2 dx,$$

subject to  $\int_0^\pi y^2(x) dx = 1, y(0) = 0 = y(\pi)$  is equal to

1. 1/2
2. 1
3. 2
4. 1/3

**JUNE – 20**

**PART – B**

28. The extremal of the functional  $J(y) = \int_0^1 [2(y')^2 + xy] dx, y(0) = 0, y(1) = 1, y \in C^2[0, 1]$
1.  $y = \frac{x^2}{12} + \frac{11x}{12}$
  2.  $y = \frac{x^3}{3} + \frac{2x^2}{3}$
  3.  $y = \frac{x^2}{7} + \frac{6x}{7}$
  4.  $y = \frac{x^3}{24} + \frac{23x}{24}$

**PART – C**

29. The extremal of the functional  $J(y) = \int_0^1 e^x \sqrt{1 + (y')^2} dx, y \in C^2[0,1]$  is of the form
1.  $y = \sec^{-1} \left( \frac{x}{c_1} \right) + c_2$ , where  $c_1$  and  $c_2$  are arbitrary constants
  2.  $y = \sec^{-1} \left( \frac{x}{c_1} \right) + c_2$ , where  $|c_1| < 1$  and  $c_2$  is an arbitrary constant
  3.  $y = \tan^{-1} \left( \frac{x}{c_1} \right) + c_2$ , where  $c_1$  and  $c_2$  are arbitrary constants
  4.  $y = \tan^{-1} \left( \frac{x}{c_1} \right) + c_2$ , where  $|c_1| > 1$  and  $c_2$  is an arbitrary constant

30. Consider the functional

$J(y) = \int_0^\pi ((y')^2 - ky^2) dx$  with boundary conditions  $y(0) = 0, y(\pi) = 0$   
Which of the following statements are true?

1. It has a unique extremal for all  $k \in \mathbb{R}$
2. It has atmost one extremal if  $\sqrt{k}$  is not an integer
3. It has infinitely many extremals if  $\sqrt{k}$  is an integer
4. It has a unique extremal if  $\sqrt{k}$  is an integer

**JUNE - 21**

**PART - B**

31. Which of the following is an extremal of the functional  $J(y) = \int_{-1}^1 (y'^2 - 2xy) dx$  that satisfy the boundary conditions  $y(-1) = -1$  and  $y(1) = 1$ ?

- |                                    |                                    |
|------------------------------------|------------------------------------|
| 1. $-\frac{x^3}{5} + \frac{6x}{5}$ | 2. $-\frac{x^5}{8} + \frac{9x}{8}$ |
| 3. $-\frac{x^3}{6} + \frac{7x}{6}$ | 4. $-\frac{x^3}{7} + \frac{8x}{7}$ |

**PART - C**

32. Let  $X = \{y \in C^1[0, \pi] : y(0) = 0 = y(\pi)\}$  and define  $J : X \rightarrow \mathbb{R}$  by

$J(y) = \int_0^\pi y^2(1 - y'^2) dx$ . Which of the following statements are true?

1.  $y \equiv 0$  is a local minimum for  $J$  with respect to the  $C^1$  norm on  $X$
2.  $y \equiv 0$  is a local maximum for  $J$  with respect to the  $C^1$  norm on  $X$
3.  $y \equiv 0$  is a local minimum for  $J$  with respect to the sup norm on  $X$
4.  $y \equiv 0$  is a local maximum for  $J$  with respect to the sup norm on  $X$

33. Let  $B$  be the unit ball in  $\mathbb{R}^3$  centered at origin. The Euler-Lagrange equation corresponding to the functional

$$I(u) = \int_B (1 + |\nabla u|^2)^{\frac{1}{2}} dx$$

$$1. \operatorname{div} \left( \frac{\nabla u}{(1 + |\nabla u|^2)^{\frac{1}{2}}} \right) = 0$$

$$2. \frac{\Delta u}{(1 + |\nabla u|^2)^{\frac{1}{2}}} = 1$$

$$3. |\nabla u| = 1$$

$$4. (1 + |\nabla u|^2) \Delta u = \sum_{i,j=1}^3 u_{x_i} u_{x_j} u_{x_i x_j}$$

**JUNE - 22**

**PART - B**

34. What is the extremal of the functional  $J[y] = \int_{-1}^0 (12xy - (y')^2) dx$  subject to  $y(0) = 0$  and  $y(-1) = 1$ ?

- |               |                               |
|---------------|-------------------------------|
| 1. $y = x^2$  | 2. $y = \frac{2x^2 + x^4}{3}$ |
| 3. $y = -x^3$ | 4. $y = \frac{x^2 + x^4}{4}$  |

**PART - C**

35. Let  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$  be two points on the  $xy$ -plane with  $x_1$  different from  $x_2$  and  $y_1 > y_2$ . Consider a curve  $C = \{z : z(x_1) = P_1, z(x_2) = P_2\}$ . Suppose that a particle is sliding down along the curve  $C$  from the point  $P_1$  to  $P_2$  under the influence of gravity. Let  $T$  be the time taken to reach point  $P_2$  and  $g$  denote the gravitational constant. Which of the following statements are true?

$$1. T = \int_{x_1}^{x_2} \sqrt{\frac{1 + (z'(x))^2}{2gz(x)}} dx$$

$$2. T = \int_{x_1}^{x_2} \frac{\sqrt{1 + (z'(x))^2}}{2gz(x)} dx$$

3.  $T$  is minimized when  $C$  is a straight line
4. The minimizer of  $T$  cannot be a straight line

36. Let  $X = \{u \in C^1[0, 1] : u(0) = u(1) = 0\}$ . Let

$$I : X \rightarrow \mathbb{R} \text{ defined as } I(u) = \int_0^1 e^{-u(t)^2} dt,$$

for all  $u \in X$ . Let  $M = \sup_{f \in X} I[f]$  and  $m = \inf_{f \in X} I[f]$ . Which of the following statements are true?

- 1.  $M = 1, m = 0$ 
2.  $1 = M > m > 0$ 
3.  $M$  is attained
4.  $m$  is attained

37. If  $y(t)$  is a stationary function of

J[y] = integral from -1 to 1 of (1-x^2)(y')^2 dx, y(-1)=1, y(1) = 1
subject to

integral from -1 to 1 of y^2 = 1.

Which of the following statements are true?

- 1. y is unique
2. y is always a polynomial of even order
3. y is always a polynomial of odd order
4. No such y exists

JUNE - 23

PART - B

38. Consider the variational problem (P)

J(y(x)) = integral from 0 to 1 of [(y')^2 - y|y|y' + xy] dx, y(0)=0, y(1)=0.

Which of the following statements is correct?

- (1) (P) has no stationary function (extremal).
(2) y = 0 is the only stationary function (extremal) for (P).
(3) (P) has a unique stationary function (extremal) y not identically equal to 0.
(4) (P) has infinitely many stationary functions (extremal).

PART - C

39. Let y(x) and z(x) be the stationary functions (extremals) of the variational problem

J(y(x), z(x)) = integral from 0 to 1 of [(y')^2 + (z')^2 + y'z'] dx

subject to y(0) = 1, y(1) = 0, z(0) = -1, z(1) = 2.

Which of the following statements are correct?

- (1) z(x) + 3y(x) = 2 for x in [0, 1].
(2) 3z(x) + y(x) = 2 for x in [0, 1].
(3) y(x) + z(x) = x for x in [0, 1].
(4) y(x) + z(x) = x for x in [0, 1].

40. Suppose y(x) is an extremal of the following functional

J(y(x)) = integral from 0 to 1 of (y(x)^2 - 4y(x)y'(x) + 4y'(x)^2) dx

subject to y(0) = 1 and y'(0) = 1/2.

Which of the following statements are true?

- (1) y is a convex function.
(2) y is concave function.
(3) y(x1 + x2) = y(x1) y(x2) for all x1, x2 in [0, 1].
(4) y(x1x2) = y(x1) + y(x2) for all x1, x2 in [0, 1].

DECEMBER - 23

PART - B

41. The cardinality of the set of extremals of

J[y] = integral from 0 to 1 of (y')^2 dx,

subject to

y(0) = 1, y(1) = 6, integral from 0 to 1 of y dx = 3

is

- (1) 0
(2) 1
(3) 2
(4) countably infinite

PART - C

42. Among the curves connecting the points (1, 2) and (2, 8), let gamma be the curve on which an extremal of the functional

J[y] = integral from 1 to 2 of (1 + x^3 y') y' dx

can be attained. Then which of the following points lie on the curve gamma?

- (1) (sqrt(2), 3) (2) (sqrt(2), 6)
(3) (sqrt(3), 22/3) (4) (sqrt(3), 23/3)

43. Define

S = {y in C^1 [0, pi] : y(0) = y(pi) = 0}

||f||\_inf = max over x in [0, pi] of |f(x)|, for all f in S

B0 (f, epsilon) = {f in S : ||f||\_inf < epsilon}

B1 (f, epsilon) = {f in S : ||f||\_inf + ||f'||\_inf < epsilon}

Consider the functional J : S -> R given by

J[y] = integral from 0 to pi of (1 - (y')^2) y^2 dx.

Then there exists epsilon > 0 such that

- (1) J[y] <= J[0], for all y in B0 (0, epsilon)



- (2)  $J[y] \leq J[0]$ , for all  $y \in B_1(0, \varepsilon)$
- (3)  $J[y] \geq J[0]$ , for all  $y \in B_0(0, \varepsilon)$
- (4)  $J[y] \geq J[0]$ , for all  $y \in B_1(0, \varepsilon)$





## ANSWERS

- |           |           |           |
|-----------|-----------|-----------|
| 1. (2)    | 2. (1,2)  | 3. (1,2)  |
| 4. (3)    | 5. (2)    | 6. (2,4)  |
| 7. (4)    | 8. (1,3)  | 9. (3)    |
| 10. (1,3) | 11. (2,4) | 12. (1)   |
| 13. (3)   | 14. (1,3) | 15. (1)   |
| 16. (1)   | 17. (1)   | 18. (1)   |
| 19.       | 20. (4)   | 21.       |
| 22. (1)   | 23. (1)   | 24. (3,4) |
| 25. (1)   | 26. (3)   | 27. (2)   |
| 28. (4)   | 29. (1)   | 30. (2,3) |
| 31. (3)   | 32. (1)   | 33. (1,4) |
| 34. (3)   | 35.       | 36. (3,4) |
| 37.       | 38. (3)   | 39. (1,3) |
| 40. (1,3) | 41. (2)   | 42. (2,3) |
| 43.       |           |           |

