REAL ANALYSIS

PREVIOUS YEAR PAPERS

DEC - 2014

PART - B

1. Let $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. Then a necessary and sufficient condition for the sequence of polynomials $f_n(x) = b_n x + c_n x^2$ to converge uniformly to 0 on the real line is

1. $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$
2. $\sum_{n=1}^{\infty} |b_n| < \infty$ and $\sum_{n=1}^{\infty} |c_n| < \infty$
3. There exists a positive integer N such that $b_n=0$ and $c_n=0$ for all n> N
4. $\lim_{n\to\infty} c_n = 0$

2. Let k be a positive integer. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} z^n$ is

1. k
2. k^{-k}
3. k^k
4. ∞

3. Suppose p is a polynomial with real coefficients. Then which of the following statements is necessarily true?

1. There is no root of the derivative p' between two real roots of the polynomial p.
2. There is exactly one root of the derivative p' between any two real roots of p.
3. There is exactly one root of the derivative p' between any two consecutive roots of p.
4. There is at least one root of the derivative p' between any two consecutive roots of p.

4. Let $G = \{(x, f(x)) : 0 \le x \le 1\}$ be the graph of a real valued differentiable function f. Assume that $(1,0) \in G$. Suppose that the tangent vector to G at any point is perpendicular to the radius vector at that point. Then which of the following is true?

1. G is the arc of an ellipse.
2. G is the arc of a circle.
3. G is a line segment.
4. G is the arc of a parabola.

5. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}$ be a differentiable function such that (Df)(x)=0 for all $x \in \Omega$. Then which of the following is true?

1. f must be a constant function.
2. f must be constant on connected components of Ω.
3. f(x)=0 or 1 for $x \in \Omega$.
4. The range of the function f is a subset of $\mathbb{Z}$.

6. Let $\{a_n : n \ge 1\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent. Let R be the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$. Then we can conclude that

1. $0 < R < 1$
2. R=1
3. $1 < R < \infty$
4. R=∞

7. Let E be a subset of $\mathbb{R}$. Then the characteristic function $\chi_E : \mathbb{R} \to \mathbb{R}$ is continuous if and only if

1. E is closed
2. E is open
3. E is both open and closed
4. E is neither open nor closed

PART - C

8. Suppose that P is a monic polynomial of degree n in one variable with real coefficients and K is a real number. Then which of the following statements is/are necessarily true?

1. If n is even and K>0, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0)=Ke^{x_0}$
2. If n is odd and K < 0, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0)=Ke^{x_0}$
3. For any natural number n and 0< K<1, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = Ke^{x_0}$
4. If n is odd and $K \in \mathbb{R}$, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = Ke^{x_0}$

9. Let $\{a_k\}$ be an unbounded, strictly increasing sequence of positive real numbers and $x_k = (a_{k+1} - a_k)/a_{k+1}$. Which of the following statements is/are correct?

1. For all $n \ge m$, $\sum_{k=m}^{n} x_k > 1 - \frac{a_m}{a_n}$
2. There exists $n \ge m$ such that $\sum_{k=m}^{n} x_k > \frac{1}{2}$
3. $\sum_{k=1}^{\infty} x_k$ converges to a finite limit

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4. $\sum_{k=1}^{\infty} x_k$ diverges to ∞

10. For a non – empty subset S and a point x in a connected metric space (X,d), let d(x,S)=inf{d(x,y): $y \in S\}$.

Which of the following statements is/are correct?

1. If S is closed and d(x,S)>0 then x is not an accumulation point of S
2. If S is open and d(x,S)>0 then x is not an accumulation point of S.
3. If S is closed and d(x,S)>0 then S does not contain x
4. If S is open and d(x,S)=0 then $x \in S$.

11. Let f be a continuously differentiable function on ℝ. Suppose that $L = \lim_{x\to\infty}(f(x) + f'(x))$ exists. If 0<L<∞, then which of the following statements is/are correct?

1. If $\lim_{x\to\infty} f'(x)$ exists, then it is 0
2. If $\lim_{x\to\infty} f(x)$ exists, then it is L
3. If $\lim_{x\to\infty} f'(x)$ exists, then $\lim_{x\to\infty} f(x) = 0$
4. If $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} f'(x) = L$

12. Let A be a subset of ℝ. Which of the following properties imply that A is compact?

1. Every continuous function f from A to ℝ is bounded.
2. Every sequence $\{x_n\}$ in A has a convergent subsequence converging to a point in A.
3. There exists a continuous function from A onto [0,1].
4. There is no one-one and continuous function from A onto (0,1).

13. Let f be a monotonically increasing function from [0,1] into [0,1]. Which of the following statements is/are true?

1. f must be continuous at all but finitely many points in [0,1].
2. f must be continuous at all but countably many points in [0,1].
3. f must be Riemann integrable.
4. f must be Lebesgue integrable.

14. Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous function. Let G= {(x,f(x): $x \in X\}$ be the graph of f. Then

1. G is homeomorphic to X
2. G is homeomorphic to ℝ
3. G is homeomorphic to $X \times \mathbb{R}$
4. G is homeomorphic to $\mathbb{R} \times X$

15. Let $X = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$ be the unit circle inside $\mathbb{R}^2$. Let $f : X \to \mathbb{R}$ be a continuous function. Then:

1. Image (f) is connected
2. Image (f) is compact
3. The given information is not sufficient to determine whether image (f) is bounded
4. f is not injective

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PART- B

16. The sum of the series $\frac{1}{1!} + \frac{1+2}{2!} + \frac{1+2+3}{3!} + ...$ equals

1. e 2. $\frac{e}{2}$ 3. $\frac{3e}{2}$ 4. $1 + \frac{e}{2}$

17. The limit $\lim_{x\to 0} \frac{1}{x}\int_x^{2x} e^{-t^2} dt$

1. does not exist.
2. is infinite.
3. exists and equals 1.
4. exists and equals 0.

18. Let $f : X \to X$ such that $f(f(x)) = x$ for all $x \in X$. Then

1. f is one to-one and onto.
2. f is one to-one ,but not onto
3. f is onto but not one-to-one.
4. f need not be either one-to-one or onto.

19. A polynomial of odd degree with real coefficients must have

1. at least one real root.
2. no real root.
3. only real roots
4. at least one root which is not real.

20. Let for each n≥1, C_n be the open disc in $\mathbb{R}^2$, with centre at the point (n,0) and radius equal to n. Then $C = \bigcup_{n\ge 1} C_n$ is

1. {(x,y) $\in \mathbb{R}^2$: x > 0 and |y| < x}
2. {(x,y) $\in \mathbb{R}^2$: x > 0 and |y| < 2x}
3. {(x,y) $\in \mathbb{R}^2$: x > 0 and |y| < 3x}
4. {(x,y) $\in \mathbb{R}^2$: x > 0 }

PART - C

21. Let a be positive real number. Which of the following integrals are convergent?

1. $\int_0^a \frac{1}{x^4}dx$ 2. $\int_0^a \frac{1}{\sqrt{x}}dx$

3. $\int_4^\infty \frac{1}{x\log_e x}dx$ 4. $\int_5^\infty \frac{1}{x(\log_e x)^2}dx$

22. For n ≥ 1, let $g_n(x)=\sin^2\left(x+\frac{1}{n}\right), x\in[0,\infty)$ and $f_n(x)=\int_0^x g_n(t)dt$. Then

1. $\{f_n\}$ converges pointwise to a function f on $[0,\infty)$, but does not converge uniformly on $[0,\infty)$.
2. $\{f_n\}$ does not converge pointwise to any function on $[0,\infty)$.
3. $\{f_n\}$ converges uniformly on [0,1].
4. $\{f_n\}$ converges uniformly on $[0,\infty)$.

23. Which of the following sets in $\mathbb{R}^2$ have positive Lebesgue measure?

[For two sets A, B $\subseteq \mathbb{R}^2$,
A + B = {a + b | a ∈ A, b ∈ B}]

1. S = {(x,y)| $x^2 + y^2 = 1$}
2. S = {(x,y)| $x^2 + y^2 < 1$}
3. S = {(x,y)| x = y} + {(x,y)| x = - y}
4. S = {(x,y)| x = y} + {(x,y)| x = y}

24. Let f be a bounded function on ℝ and $a\in\mathbb{R}$. For δ > 0, Let ω(a, δ) = sup|f(x)-f(1)|, x∈[a-δ, a+δ]. Then

1. $\omega(a,\delta_1) \le \omega(a,\delta_2)$ if $\delta_1 \le \delta_2$
2. $\lim_{\delta\to0+}\omega(a,\delta)=0$ *for all* $a\in\mathbb{R}$
3. $\lim_{\delta\to0+}\omega(a,\delta)$ need not exist.
4. $\lim_{\delta\to0+}\omega(a,\delta)=0$ if and only if f is continuous at a.

25. For n ≥ 2, let $a_n=\frac{1}{n\log n}$. Then

1. The sequence $\{a_n\}_{n=2}^\infty$ is convergent.
2. The series $\sum_{n=2}^\infty a_n$ is convergent.
3. The series $\sum_{n=2}^\infty a_n^2$ is convergent.
4. The series $\sum_{n=2}^\infty (-1)^n a_n$ is convergent.

26. Which of the following sets of functions are uncountable? (ℕ stands for the set of natural numbers.)

1. {f|f: ℕ→ {1,2}}
2. {f|f: {1,2}→ ℕ}
3. {f|f: {1,2}→ ℕ, f(1) ≤ f(2)}
4. {f|f: ℕ→ {1,2}, f(1) ≤ f(2)}

27. Let $\{a_0,a_1,a_2,\ldots.\}$ be a sequence of real numbers. For any k ≥ 1, let $S_n=\sum_{k=0}^n a_{2k}$. Which of the following statements are correct?

1. If $\lim_{n\to\infty} S_n$ exists, then $\sum_{m=0}^\infty a_m$ exits.
2. If $\lim_{n\to\infty} S_n$ exists, then $\sum_{m=0}^\infty a_m$ need not exist.
3. If $\sum_{m=0}^\infty a_m$ exists, then $\lim_{n\to\infty} S_n$ exists.
4. If $\sum_{m=0}^\infty a_m$ exists, then $\lim_{n\to\infty} S_n$ need not exist.

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PART – B

28. For (x,y) ∈ $\mathbb{R}^2$ with (x, y) ≠ (0,0), let θ = θ (x,y) be the unique real number such that -π < θ ≤ π and (x,y) = (r cos θ, r sin θ), where $r=\sqrt{x^2+y^2}$. Then the resulting function θ : $\mathbb{R}^2$\{(0,0)} → ℝ is

1. differentiable
2. continuous, but not differentiable
3. bounded, but not continuous
4. neither bounded, nor continuous

29. Let f : ℝ → ℝ be a twice continuously differentiable function, with f(0)= f(1) =f′(0) = 0. Then

1. f″ is the zero function.
2. f″(0) is zero.
3. f″(x) = 0 for some x ∈ (0,1)
4. f″ never vanishes.

30. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{6}}+\ldots+\frac{1}{\sqrt{2n}+\sqrt{2n+2}}\right)$ is

1. $\sqrt{2}$ 2. $\frac{1}{\sqrt{2}}$

3. $\sqrt{2}+1$ 4. $\frac{1}{\sqrt{2}+1}$

31. Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$. Which of the following is true?

1. $S_{2^n} \geq \frac{n}{2}$ for every $n \geq 1$.
2. S_n is a bounded sequence
3. $\left|S_{2^n} - S_{2^{n-1}}\right| \to 0$ as $n \to \infty$
4. $\frac{S_n}{n} \to 1$ as $n \to \infty$

32. Let A be a closed subset of ℝ, A ≠ ϕ, A ≠ ℝ. Then A is:
1. the closure of the interior of A.
2. a countable set
3. a compact set
4. not open

33. Let f : [0, ∞) → [0, ∞) be a continuous function. Which of the following is correct
1. There is $x_0 \in [0, \infty)$ such that $f(x_0) = x_0$
2. If $f(x) \leq M$ for all $x \in [0, \infty)$ for some $M > 0$, then there exists $x_0 \in [0, \infty)$ such that $f(x_0) = x_0$
3. If f has a fixed point, then it must be unique
4. f does not have a fixed point unless it is differentiable on (0, ∞)

PART – C

34. Let f : ℝ → ℝ be a differentiable function such that $\sup_{x\in\mathbb{R}} |f'(x)| < \infty$. Then,
1. f maps a bounded sequence to a bounded sequence.
2. f maps a Cauchy sequence to a Cauchy sequence.
3. f maps a convergent sequence to a convergent sequence.
4. f is uniformly continuous.

35. For $(x,y) \in \mathbb{R}^2$, consider the series $\lim\limits_{n\to\infty} \sum_{\ell, k=0}^{n} \frac{k^2 x^k y^\ell}{\ell!}$. Then the series converges for (x,y) in

1. (-1, 1) x (0, ∞)
2. ℝ x (-1, 1)
3. (-1, 1) x (-1, 1)
4. ℝ x ℝ

36. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the formula $f(x,y) = (3x + 2y + y^2 + |xy|, 2x + 3y + x^2 + |xy|)$. Then,
1. f is discontinuous at (0,0).
2. f is continuous at (0,0) but not differentiable at (0,0).
3. f is differentiable at (0,0).
4. f is differentiable at (0,0) and the derivative Df (0,0) is invertible.

37. Let $p_n(x) = a_n x^2 + b_n x$ be a sequence of quadratic polynomials where $a_n, b_n \in \mathbb{R}$ for all $n \geq 1$. Let λ_0, λ_1 be distinct nonzero real numbers such that $\lim_{n\to\infty} p_n(\lambda_0)$ and $\lim_{n\to\infty} p_n(\lambda_1)$ exist. Then,
1. $\lim_{n\to\infty} p_n(x)$ exists for all $x \in \mathbb{R}$.
2. $\lim_{n\to\infty} p'_n(x)$ exists for all $x \in \mathbb{R}$.
3. $\lim_{n\to\infty} p_n\left(\frac{\lambda_0+\lambda_1}{2}\right)$ does not exist.
4. $\lim_{n\to\infty} p'_n\left(\frac{\lambda_0+\lambda_1}{2}\right)$ does not exist.

38. Let t and a be positive real numbers. Define $B_a = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leq a^2\}$. Then for any compactly supported continuous function f on $\mathbb{R}^n$ which of the following are correct?
1. $\int_{B_a} f(tx)\,dx = \int_{B_{ta}} f(x)\,t^{-n}dx$
2. $\int_{B_a} f(tx)\,dx = \int_{B_{t^n a}} f(x)\,t\,dx$
3. $\int_{R^n} f(x+y)\,dx = \int_{R^n} f(x)\,dx,$ for some $y \in \mathbb{R}^n$.
4. $\int_{R^n} f(tx)\,dx = \int_{R^n} f(x)\,t^n dx.$

39. Consider all sequences $\{f_n\}$ of real valued continuous functions on [0, ∞). Identify which of the following statements are correct.
1. If $\{f_n\}$ converges to f pointwise on [0, ∞), then $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx$
2. If $\{f_n\}$ converges to f uniformly on [0, ∞), then $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx$
3. If $\{f_n\}$ converges to f uniformly on [0, ∞), then f is continuous on [0, ∞).
4. There exists a sequence of continuous functions $\{f_n\}$ on [0, ∞) such that $\{f_n\}$ converges to f uniformly on [0, ∞) but $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq \int_0^\infty f(x)\,dx.$

40. Let G_1 and G_2 be two subsets of $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Then,
1. $f^{-1}(G_1 \cup G_2) = f^{-1}(G_1) \cup f^{-1}(G_2)$
2. $f^{-1}(G_1^c) = (f^{-1}(G_1))^c$
3. $f(G_1 \cap G_2) = f(G_1) \cap f(G_2)$

4. If G_1 is open and G_2 is closed then $G_1 + G_2 = \{x + y : x \in G_1, y \in G_2\}$ is neither open nor closed.

41. Let $A = \{(x,y) \in \mathbb{R}^2 : x+y \neq -1\}$. Define $f:A \to \mathbb{R}^2$ by $f(x,y) = \left(\frac{y}{1+x+y}, \frac{x}{1+x+y}\right)$. Then,
1. the determinant of the Jacobian of f does not vanish on A.
2. f is infinitely differentiable on A.
3. f is one to one.
4. $f(1) = \mathbb{R}^2$.

42. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(r, \theta) = (r \cos\theta, r \sin\theta)$. Then for which of the open subsets U of $\mathbb{R}^2$ given below, f restricted to U admits an inverse?
1. $U = \mathbb{R}^2$
2. $U = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$
3. $U = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1$
4. $U = \{(x,y) \in \mathbb{R}^2 : x < -1, y < -1\}$

43. Let $S \subset \mathbb{R}^2$ be defined by
$S = \{ \left(m + \frac{1}{4^{|p|}}, n + \frac{1}{4^{|q|}}\right) : m, n, p, q \in \mathbb{Z}\}$.
Then,
1. S is discrete in $\mathbb{R}^2$.
2. The set of limit points of S is the set $\{(m,n) : m,n \in \mathbb{Z}\}$.
3. S^C is connected but not path connected.
4. S^C is path connected.

44. Which of the following statements is/are true?
1. There exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{Q}$.
2. There exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{Z}$.
3. There exists a continuous map $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\mathbb{R}) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
4. There exists a continuous map $f: [0,1] \cup [2,3] \to \{0,1\}$.

45. Let $f : (0, 1) \to \mathbb{R}$ be continuous. Suppose that $|f(x) - f(y)| \leq |\cos x - \cos y|$ for all $x,y \in (0,1)$. Then,
1. f is discontinuous at least at one point in (0, 1).
2. f is continuous everywhere on (0, 1) but not uniformly continuous on (0, 1).
3. f is uniformly continuous on (0, 1).
4. $\lim_{x\to 0} f(x)$ exists.

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PART – B

46. Consider the improper Riemann integral $\int_0^x y^{-1/2} dy$. This integral is:
1. continuous in $[0, \infty)$
2. continuous only in $(0, \infty)$
3. discontinuous in $(0, \infty)$
4. discontinuous only in $(1/2, \infty)$

47. Which one of the following statements is true for the sequence of functions
$f_n(x) = \frac{1}{n^2 + x^2}, n = 1,2,..., x \in [1/2,1]$?
1. The sequence is monotonic and has 0 as the limit for all $x \in [1/2,1]$ as $n \to \infty$
2. The sequence is not monotonic but has $f(x) = \frac{1}{x^2}$ as the limit as $n \to \infty$
3. The sequence is monotonic and has $f(x) = \frac{1}{x^2}$ as the limit as $n \to \infty$
4. The sequence is not monotonic but has 0 as the limit

48. $\lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right)^n$ equals

1. 1
2. $e^{-1/2}$
3. e^{-2}
4. e^{-1}

49. Consider the interval (-1,1) and a sequence $\{\alpha_n\}_{n-1}^{\infty}$ of elements in it. Then,
1. Every limit point of $\{\alpha_n\}$ is in (-1,1)
2. Every limit point of $\{\alpha_n\}$ is in [-1,1]
3. The limit points of $\{\alpha_n\}$ can only be in {-1,0,1}
4. The limit points of $\{\alpha_n\}$ cannot be in {-1,0,1}

50. Let F:R→R be a mnonotonic function. Then
1. F has no discontinuities
2. F has only finitely many discontinuities
3. F can have at most countably many discontinuities
4. F can have uncountably many discontinuities

51. Consider the function

$f\{x,y\} = \frac{x^2}{y^2}, (x,y) \in [1/2, 3/2] \times [1/2, 3/2]$.

The derivative of the function at (1,1) along the direction (1,1) is

1. 0 2. 1
3. 2 4. -2

PART – C

52. Let $\{x_n\}$ be an arbitrary sequence of real numbers. Then

1. $\sum_{n=1}^{\infty} |x_n|^p < \infty$ for some $1 < p < \infty$ implies $\sum_{n=1}^{\infty} |x_n|^q < \infty$ for q > p.
2. $\sum_{n=1}^{\infty} |x_n|^p < \infty$ for some $1 < p < \infty$ implies $\sum_{n=1}^{\infty} |x_n|^q < \infty$ for any 1 ≤ q < p.
3. Given any $1 < p < q < \infty$, there is a real sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ but $\sum_{n=1}^{\infty} |x_n|^q = \infty$.
4. Given any $1 < q < p < \infty$, there is a real sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ but $\sum_{n=1}^{\infty} |x_n|^q = \infty$.

53. Let f : ℝ → ℝ be a continuous function and f(x+1) = f(x) for all x∈ℝ. Then

1. f is bounded above, but not bounded below
2. f is bounded above and below, but may not attain its bounds.
3. f is bounded above and below and f attains its bounds.
4. f is uniformly continuous.

54. Let $x_1 = 0$, $x_2 = 1$, and for n ≥ 3, define $x_n = \frac{x_{n-1} + x_{n-2}}{2}$. Which of the following is/are true?

1. $\{x_n\}$ is a monotone sequence.
2. $\lim_{n\to\infty} x_n = \frac{1}{2}$.
3. $\{x_n\}$ is a Cauchy sequence.
4. $\lim_{n\to\infty} x_n = \frac{2}{3}$.

55. Take the closed interval [0,1] and open interval (1/3, 2/3). Let K = [0,1]\(1/3,2/3). For x∈[0,1] define f(x) = d(x,K) where d(x,K) = inf{|x – y| | y ∈ K}. Then

1. f : [0,1] →ℝ is differentiable at all points of (0,1)
2. f: [0,1] →ℝ is not differentiable at 1/3 and 2/3
3. f: [0,1] →ℝ is not differentiable at 1/2
4. f: [0,1] →ℝ is not continuous

56. Which of the following functions is/are uniformly continuous on the interval (0,1) ?

1. $\frac{1}{x}$ 2. $\sin\frac{1}{x}$
3. $x\sin\frac{1}{x}$ 4. $\frac{\sin x}{x}$

57. Let A be any set. Let ℙ(1) be the power set of A, that is, the set of all subsets of A; ℙ(1) = {B : B ⊆A}.
Then which of the following is/are true about the set ℙ(1)?

1. ℙ(1)= ϕ for some A.
2. ℙ(1) is a finite set for some A.
3. ℙ(1) is a countable set for some A.
4. ℙ(1) is a uncountable set for some A.

58. Define f on [0,1] by

$$f(x) = \begin{cases} x^2 \text{ if } x \text{ is rational} \\ x^3 \text{ if } x \text{ is irrational} \end{cases}$$. Then

1. f is not Riemann integrable on [0,1].
2. f is Riemann integrable and $\int_0^1 f(x)dx = \frac{1}{4}$.
3. f is Riemann integrable and $\int_0^1 f(x)dx = \frac{1}{3}$.
4. $\frac{1}{4} = \underline{\int_0^1} f(x)dx < \overline{\int_0^1} f(x)dx = \frac{1}{3}$, where $\underline{\int_0^1} f(x)dx$ and $\overline{\int_0^1} f(x)dx$ are the lower and upper Riemann integrals of f.

59. Consider the integral $A = \int_0^1 x^n (1-x)^n dx$.

Pick each correct statement from below.

1. A is not a rational number
2. $0 < A \le 4^{-n}$.
3. A is a natural number.
4. A^{-1} is a natural number.

DEC-2016

PART – B

60. Consider the sets of sequences

$X=\{(x_n) : x_n \in \{0,1\}, n \in \mathbb{N}\}$ and
$Y=\{(x_n) \in X : x_n = 1$ for at most finitely many n$\}$.
Then

1. X is countable, Y is finite.
2. X is uncountable, Y is countable.
3. X is countable, Y is countable.
4. X is uncountable, Y is uncountable.

61. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x^2, y^2 + \sin x)$. Then the derivative of f at (x, y) is the linear transformation given by

1. $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$
2. $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$
3. $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$
4. $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

62. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = xy$. Let $v = (1, 2)$ and $a = (a_1, a_2)$ be two elements of $\mathbb{R}^2$. The directional derivative of f in the direction of v at a is:

1. $a_1 + 2a_2$
2. $a_2 + 2a_1$
3. $\frac{a_2}{2} + a_1$
4. $\frac{a_1}{2} + a_2$

63. $\lim_{n\to\infty} \frac{1}{n^4} \sum_{j=0}^{2n-1} j^3$ equals

1. 4
2. 16
3. 1
4. 8

64. $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 0$ and $\left|\frac{df}{dx}(x)\right| \le 5$ for all x. We can conclude that f(1) is in

1. (5, 6)
2. [-5, 5]
3. $(-\infty, -5) \cup (5, \infty)$
4. [-4, 4]

65. Let G be an open set in $\mathbb{R}^n$. Two points $x, y \in G$ are said to be equivalent if they can be joined by a continuous path completely lying inside G. Number of equivalence classes is

1. only one
2. at most finite
3. at most countable
4. can be finite, countable or uncountable

PART – C

66. Let $s \in (0,1)$. Then decide which of the following are true.

1. $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. $s > m/n$
2. $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. $s < m/n$
3. $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. $s = m/n$
4. $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. $s = m+n$

67. Let $f_n(x)=(-x)^n$, $x \in [0,1]$. Then decide which of the following are true.

1. there exists a pointwise convergent subsequence of f_n.
2. f_n has no pointwise convergent subsequence.
3. f_n converges point wise everywhere.
4. f_n has exactly one point wise convergent subsequence.

68. Which of the following are true for the function $f(x)=\sin(x)\sin\left(\frac{1}{x}\right)$, $x \in (0,1)$?

1. $\underline{\lim}_{x\to 0} f(x) = \overline{\lim}_{x\to 0} f(x)$
2. $\underline{\lim}_{x\to 0} f(x) < \overline{\lim}_{x\to 0} f(x)$
3. $\underline{\lim}_{x\to 0} f(x) = 1$
4. $\overline{\lim}_{x\to 0} f(x) = 0$

69. Find out which of the following series converge uniformly for $x \in (-\pi, \pi)$.

1. $\sum_{n=1}^{\infty} \frac{e^{-n|x|}}{n^3}$
2. $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n^5}$
3. $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$
4. $\sum_{n=1}^{\infty} \frac{1}{((x+\pi)n)^2}$

70. Decide which of the following functions are uniformly continuous on (0,1).

1. $f(x) = e^x$
2. $f(x) = x$
3. $f(x) = \tan\left(\frac{\pi x}{2}\right)$
4. $f(x) = \sin(x)$

71. Let $\chi_A(x)$ denote the function which is 1 if $x \in A$ and 0 otherwise. Consider $f(x) = \sum_{n=1}^{200} \frac{1}{n^6} \chi_{\left[0, \frac{n}{200}\right]}(x), x \in [0,1]$. Then f(x) is

1. Riemann integrable on [0,1].
2. Lebesgue integrable on [0,1]
3. is a continuous function on [0,1].
4. is a monotone function on [0,1].

72. A function f(x,y) on $\mathbb{R}^2$ has the following partial derivatives $\frac{\partial f}{\partial x}(x,y)=x^2, \frac{\partial f}{\partial y}(x,y)=y^2$.

Then
1. f has directional derivatives in all directions everywhere.
2. f has derivative at all points.
3. f has directional derivative only along the direction (1,1) everywhere.
4. f does not have directional derivatives in any direction everywhere.

73. Let d_1, d_2 be the following metrices on $\mathbb{R}^n$.
$d_1(x,y)=\sum_{i=1}^{n}|x_i-y_i|,\ d_2(x,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}}$.

Then decide which of the following is a metric on $\mathbb{R}^n$.
1. $d(x,y)=\frac{d_1(x,y)+d_2(x,y)}{1+d_1(x,y)+d_2(x,y)}$
2. $d(x,y)=d_1(x,y)-d_2(x,y)$
3. $d(x,y)=d_1(x,y)+d_2(x,y)$
4. $d(x,y)=e^{\pi}d_1(x,y)+e^{-\pi}d_2(x,y)$

74. Let A be the following subset of $\mathbb{R}^2$:
A={(x,y):$(x+1)^2+y^2\le 1$}∪{(x,y):y=xsin$\frac{1}{x}, x>0$}.

Then

1. A is connected
2. A is compact
3. A is path connected
4. A is bounded

JUNE - 2017

PART – B

75. $L=\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}$. Then

1. $L=0$
2. $L=1$
3. $0<L<\infty$
4. $L=\infty$

76. Consider the sequence $a_n=\left(1+(-1)^n\frac{1}{n}\right)^n$.

Then
1. $\limsup_{n\to\infty} a_n=\liminf_{n\to\infty} a_n=1$
2. $\limsup_{n\to\infty} a_n=\liminf_{n\to\infty} a_n=e$
3. $\limsup_{n\to\infty} a_n=\liminf_{n\to\infty} a_n=\frac{1}{e}$
4. $\limsup_{n\to\infty} a_n=e, \liminf_{n\to\infty} a_n=\frac{1}{e}$

77. For $a>0$, the series $\sum_{n=1}^{\infty}a^{\ell n n}$ is convergent if and only if

1. $0<a<e$
2. $0<a\le e$
3. $0<a<\frac{1}{e}$
4. $0<a\le\frac{1}{e}$

78. Let f : ℝ → ℝ be defined by
$$f(x)=\begin{cases}\frac{\sin x}{x} & if \quad x\neq 0\\ 1 & if \quad x=0\end{cases}$$
Then
1. f is not continuous
2. f is continuous but not differentiable
3. f is differentiable
4. f is not bounded

79. Let A = {n ∈ ℕ : n = 1 or the only prime factors of n are 2 or 3}, for example, 6 ∈ A, 10 ∉ A. Let $S=\sum_{n\in A}\frac{1}{n}$. Then

1. S is finite
2. S is a divergent series
3. S = 3
4. S = 6

80. For n ≥ 1, let $f_n(x)=xe^{-nx^2}$, x ∈ ℝ.
Then the sequence $\{f_n\}$ is
1. uniformly convergent on ℝ
2. uniformly convergent only on compact subsets of ℝ
3. bounded and not uniformly convergent on ℝ
4. a sequence of unbounded functions

81. Let $\alpha=0.101101110111110...$ be a given real number written in base 10, that is, the n-th digit of α is 1, unless n is of the form $\frac{k(k+1)}{2}-1$ in which case it is 0. Choose all the correct statements from below.
1. α is a rational number
2. α is an irrational number
3. For every integer q≥2, there exists an integer r ≥ 1 such that $\frac{r}{q}<\alpha<\frac{r+1}{q}$.
4. α has no periodic decimal expansion.

82. For a,b∈ℕ, consider the sequence

$$d_n = \frac{\binom{n}{a}}{\binom{n}{b}}$$ for n>a,b. Which of the following statements are true ? As n →∞,

1. {d_n} converges for all values of a and b
2. {d_n} converges if a < b
3. {d_n} converges if a = b
4. {d_n} converges if a > b

83. Let {a_n} be a sequence of real numbers satisfying $\sum_{n=1}^{\infty}|a_n - a_{n-1}| < \infty$. Then the series $\sum_{n=0}^{\infty} a_n x^n, x \in \mathbb{R}$ is convergent

1. nowhere on ℝ
2. everywhere on ℝ
3. on some set containing (-1,1)
4. only on (-1,1)

84. Let $f(x) = \tan^{-1} x, x \in \mathbb{R}$. Then

1. there exists a polynomial p(x) satisfies $p(x)f'(x) = 1$, for all x
2. $f^{(n)}(0) = 0$ for all positive even integer n
3. The sequence $\{f^{(n)}(0)\}$ is unbounded
4. $f^{(n)}(0) = 0$ for all n

85. Let $f_n(x) = \frac{1}{1+n^2x^2}$ for n∈ℕ, x∈ℝ. Which of the following are true?

1. f_n converges pointwise on [0,1] to a continuous function
2. f_n converges uniformly on [0,1]
3. f_n converges uniformly on $\left[\frac{1}{2}, 1\right]$
4. $\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 (\lim_{n\to\infty} f_n(x))dx$

86. If $\lambda_n = \int_0^1 \frac{dt}{(1+t)^n}$ for n∈ℕ, then

1. λ_n does not exist for some n
2. λ_n exists for every n and the sequence is unbounded
3. λ_n exists for every n and the sequence is bounded
4. $\lim_{n\to\infty}(\lambda_n)^{1/n} = 1$

87. The equation $11^x + 13^x + 17^x - 19^x = 0$ has

1. no real root
2. only one real root
3. exactly two real roots
4. more than two roots

88. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(\underline{x}) = a_1x_1^2 + a_2x_2^2 + ... + a_nx_n^2$, where $\underline{x} = (x_1, x_2, ..., x_n)$ and at least one a_j is not zero. Then we can conclude that

1. f is not everywhere differentiable
2. the gradient $(\nabla f)(\underline{x}) \neq 0$ for every $\underline{x} \in \mathbb{R}^n$
3. If $\underline{x} \in \mathbb{R}^n$ is such that $(\nabla f)(\underline{x}) = 0$ then $f(\underline{x}) = 0$
4. If $x \in \mathbb{R}^n$ is such that $f(\underline{x}) = 0$ then $(\nabla f)(\underline{x}) = 0$

89. Let S be the set of (α,β)∈ℝ² such that $\frac{x^\alpha y^\beta}{\sqrt{x^2+y^2}} \to 0$ as (x,y) → (0,0). Then S is contained in

1. $\{(\alpha,\beta) : \alpha > 0, \beta > 0\}$
2. $\{(\alpha,\beta) : \alpha > 2, \beta > 2\}$
3. $\{(\alpha,\beta) : \alpha + \beta > 1\}$
4. $\{(\alpha,\beta) : \alpha + 4\beta > 1\}$

DECEMBER 2017

PART - B

90. Let ℤ denote the set of integers and $\mathbb{Z}_{20}$ denote the set (0, 1, 2, 3,…). Consider the map f : $\mathbb{Z}_{20} \times \mathbb{Z} \to \mathbb{Z}$ given by f(m, n) = 2^m. (2n + 1). Then the map f is

1. onto (surjective) but not one-one (injective)
2. one-one (injective) but not onto (surjective)
3. both one-one and onto
4. neither one-one nor onto

91. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers satisfying $a_1 \geq 1$ and $a_{n+1} \geq a_n + 1$ for all n ≥ 1. Then which of the following is necessarily true?

1. The series $\sum_{n=1}^{\infty}\frac{1}{a_n^2}$ diverges

2. The sequence $\{a_n\}_{n\geq 1}$ is bounded

3. The series $\sum_{n=1}^{\infty}\frac{1}{a_n^2}$ converges

4. The series $\sum_{n=1}^{\infty}\frac{1}{a_n}$ converges

92. Let D be a subset of the real line. Consider the assertion: "Every infinite sequence in D has a subsequence which converges in D". This assertion is true if

1. D = [0, ∞)
2. D = [0, 1] ∪ [3, 4]
3. D = [-1, 1) ∪ (1, 2]
4. D = (-1, 1]

93. Let f : (0, ∞) → ℝ be uniformly continuous. Then

1. $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$ exist
2. $\lim_{x\to 0^+} f(x)$ exits but $\lim_{x\to\infty} f(x)$ need not exist
3. $\lim_{x\to 0^+} f(x)$ need not exist but $\lim_{x\to\infty} f(x)$ exists
4. neither $\lim_{x\to 0^+} f(x)$ nor $\lim_{x\to\infty} f(x)$ need exist

94. Let S = {f : ℝ → ℝ| ∃ ϵ > 0 such that ∀ δ > 0, |x – y| < δ ⇒ |f(x) – f(y)| < ϵ}. Then

1. S = {f : ℝ → ℝ | f is continuous}
2. S = {f : ℝ → ℝ | f is uniformly continuous}
3. S = {f : ℝ → ℝ | f is bounded}
4. S = {f : ℝ → ℝ | f is constant}

95. Which of the following is necessarily true for a function f : X → Y?

1. if f is injective, then there exists g : Y → X such that f(g(y)) = y for all y ∈ Y
2. if f is surjective, then there exists g :Y → X such that f(g(y)) = y for all y ∈ Y
3. if f is injective and Y is countable then X is finite
4. if f is surjective and X is uncountable then Y is countably infinite

96. Let k be a positive integer and let S_k = {x ∈ [0, 1] | a decimal expansion of x has a prime digit at its k^{th} place}. Then the Lebesgue measure of S_k is

1. 0
2. 4/10
3. $(4/10)^k$
4. 1

97. Let S = {x ∈ [-1, 4] | sin (x) > 0}. Which of the following is true?

1. inf (S) < 0
2. sup (S) does not exist
3. sup (S) = π
4. inf (S) = π/2

PART – C

98. Which of the following are convergent?

1. $\sum_{n=1}^{\infty} n^2 2^{-n}$
2. $\sum_{n=1}^{\infty} n^{-2} 2^{n}$
3. $\sum_{n=2}^{\infty} \frac{1}{n\log n}$
4. $\sum_{n=1}^{\infty} \frac{1}{n\log(1+1/n)}$

99. Let $a_{mn}, m\geq 1, n\geq 1$ be a double array of real numbers. Define

$$P=\liminf_{n\to\infty}\liminf_{m\to\infty} a_{mn},$$
$$Q=\liminf_{n\to\infty}\limsup_{m\to\infty} a_{mn},$$
$$R=\limsup_{n\to\infty}\liminf_{m\to\infty} a_{mn},$$
$$S=\limsup_{n\to\infty}\limsup_{m\to\infty} a_{mn}$$

Which of the following statements are necessarily true ?

1. P≤Q
2. Q≤R
3. R≤S
4. P≤S

100. Let ℝ denote the set of real numbers and ℚ the set of all rational numbers. For $0\leq\in\leq\frac{1}{2}$, let $A_\in$ be the open interval (0,1-∈). Which of the following are true?

1. $\sup_{0<\in<\frac{1}{2}} \sup(A_\in)<1$
2. $0<\in_1<\in_2<\frac{1}{2}\Rightarrow \inf(A_{\in_1})<\inf(A_{\in_2})$
3. $0<\in_1<\in_2<\frac{1}{2}\Rightarrow \sup(A_{\in_1})>\sup(A_{\in_2})$
4. $\sup(A_\in\cap Q)=\sup(A_\in\cap(R\setminus Q))$

101. Let f: $\mathbb{R}\to\mathbb{R}$ be a function satisfying f(x+y) = f(x)f(y), $\forall$ x,y$\in\mathbb{R}$ and $\lim_{x\to 0} f(x)=1$. Which of the following are necessarily true ?

1. f is strictly increasing
2. f is either constant or bounded
3. $f(rx)=f(x)^r$ for every rational $r\in\mathbb{Q}$
4. f(x) $\geq$ 0, $\forall$ x$\in\mathbb{R}$

102. Consider the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the usual metric. Suppose a and b are irrational numbers with a < b and let K=[a,b] $\cap\mathbb{Q}$. Then

1. K is a bounded subset of $\mathbb{Q}$
2. K is a closed subset of $\mathbb{Q}$
3. K is a compact subset of $\mathbb{Q}$
4. K is an open subset of $\mathbb{Q}$

103. Evaluate $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{n}{k^2+n^2}$

1. $\frac{\pi}{2}$ 2. π 3. $\frac{\pi}{8}$ 4. $\frac{\pi}{4}$

104. Let $f(x,y)=\frac{1-\cos(x+y)}{x^2+y^2}$ if (x,y) $\neq$ (0,0)

$f(0,0)=\frac{1}{2}$ and

$g(x,y)=\frac{1-\cos(x+y)}{(x+y)^2}$ if $x+y\neq 0$

$g(x,y)=\frac{1}{2}$ if $x+y=0$

Then

1. f is continuous at (0,0)
2. f is continuous everywhere except at (0,0)
3. g is continuous at (0,0)
4. g is continuous everywhere

105. Let f: $\mathbb{R}^4\to\mathbb{R}$ be defined by f(x) = x^t Ax, where A is a 4×4 matrix with real entries and x^t denotes the transpose of x. The gradient of f at a point x_0 necessarily is

1. $2Ax_0$
2. $Ax_0 + A^t x_a$
3. $2A^t x_0$
4. Ax_0

106. Let f: $\mathbb{R}^n\to\mathbb{R}^n$ be a continuously differentiable map satisfying

$\|f(x) - f(y)\| \geq \|x-y\|$, for all x,y$\in\mathbb{R}^n$. Then

1. f is onto
2. f($\mathbb{R}^n$) is a closed subset of $\mathbb{R}^n$
3. f($\mathbb{R}^n$) is an open subset of $\mathbb{R}^n$
4. f(0) = 0

107. Consider $X=\left\{\left(x,\sin\frac{1}{x}\right)\middle| 0<x\leq 1\right\}\cup$ $\{(0,y)\mid -1\leq y\leq 1\}$ as a subspace of $\mathbb{R}^2$ and Y=[0,1) as a subspace of $\mathbb{R}$. Then

1. X is connected
2. X is compact
3. X ×Y (in product topology) is connected
4. X×Y (in product topology) is compact

108. Let $l^2=\{x=(x_n)_{n\geq 1}\mid x_n\in\mathbb{R},$ $\sum_{n=1}^{\infty}x_n^2<\infty\}$ be the Hilbert space of square summable sequences and let e_k denote the kth co-ordinate vector (with 1 in kth place, 0 elsewhere). Which of the following subspaces is NOT dense in ℓ^2?

1. span $\{e_1-e_2, e_2-e_3, e_3-e_4,...\}$
2. span $\{2e_1-e_2, 2e_2-e_3, 2e_3-e_4,...\}$
3. span $\{e_1-2e_2, e_2-2e_3, e_3-2e_4,...\}$
4. span $\{e_2, e_3, e_4,...\}$

109. Let f : [-1,1] $\to\mathbb{R}$ be a function given by

$f(x)=\begin{cases} x^2\cos\left(\frac{1}{x}\right) & if \quad x\neq 0 \\ 0 & if \quad x=0\end{cases}$. Then

1. f is of bounded variation on [-1,1]
2. f′ is of bounded variation on [-1,1]
3. |f′(x)| ≤ 1 $\forall$ x$\in$[-1,1]
4. |f′(x)| ≤ 3 $\forall$ x$\in$[-1,1]

JUNE - 2018

PART – B

110. Given that there are real constants a,b,c,d such that the identity

$\lambda x^2+2xy+y^2=(ax+by)^2+(cx+dy)^2$

holds for all x,y$\in\mathbb{R}$. This implies

1. $\lambda=-5$
2. $\lambda\geq 1$
3. 0 < $\lambda<1$
4. there is no such $\lambda\in\mathbb{R}$

111. Given $\{a_n\}$, $\{b_n\}$ two monotone sequences of real numbers and that $\sum a_n b_n$ is convergent, which of the following is true?
1. $\sum a_n$ is convergent and $\sum b_n$ is convergent
2. At least one of $\sum a_n$, $\sum b_n$ is convergent
3. $\{a_n\}$ is bounded and $\{b_n\}$ is bounded
4. At least one of $\{a_n\}$, $\{b_n\}$ is bounded

112. Let $S = \{(x, y) \mid x^2 + y^2 = \frac{1}{n^2},$ where $n \in \mathbb{N}$ and either $x \in \mathbb{Q}$ or $y \in \mathbb{Q}\}$. Here $\mathbb{Q}$ is the set of rational numbers and $\mathbb{N}$ is the set of positive integers. Which of the following is true?
1. S is a finite non empty set
2. S is countable
3. S is uncountable
4. S is empty

113. Define the sequence $\{a_n\}$ as follows:
$a_1 = 1$ and for $n \geq 1$,
$a_{n+1} = (-1)^n \left(\frac{1}{2}\right)\left(|a_n| + \frac{2}{|a_n|}\right)$. Which of the following is true?
1. $\limsup a_n = \sqrt{2}$
2. $\liminf a_n = -\infty$
3. $\lim a_n = \sqrt{2}$
4. $\sup a_n = \sqrt{2}$

114. If $\{x_n\}$ is a convergent sequence in $\mathbb{R}$ and $\{y_n\}$ is a bounded sequence in $\mathbb{R}$, then we can conclude that
1. $\{x_n + y_n\}$ is convergent
2. $\{x_n + y_n\}$ is bounded
3. $\{x_n + y_n\}$ has no convergent subsequence
4. $\{x_n + y_n\}$ has no bounded subsequence

115. The difference $\log(2) - \sum_{n=1}^{100} \frac{1}{2^n \cdot n}$ is
1. less than 0
2. greater than 1
3. less than $\frac{1}{2^{100} \cdot 101}$
4. greater than $\frac{1}{2^{100} \cdot 101}$

116. Let $f(x, y) = \log(\cos^2(e^{x^2})) + \sin(x + y)$.
Then $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y)$ is
1. $\frac{\cos(e^{x^2}) - 1}{1 + \sin^2(e^{x^2})} - \cos(x + y)$
2. 0
3. $-\sin(x + y)$
4. $\cos(x + y)$

117. Let $f(x) = x^5 - 5x + 2.$ Then
1. f has no real root
2. f has exactly one real root
3. f has exactly three real roots
4. all roots of f are real

118. Consider the space $S = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{Q}\} \subset \mathbb{R}^2$, where $\mathbb{Q}$ is the set of rational numbers. Then
1. S is connected in $\mathbb{R}^2$
2. S^c is connected in $\mathbb{R}^2$
3. S is closed in $\mathbb{R}^2$
4. S^c is closed in $\mathbb{R}^2$

PART-C

119. For each $\alpha \in \mathbb{R}$,
let $S_\alpha = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \alpha^2\}$.
Let $E = \bigcup_{\alpha \in \mathbb{R} \setminus \mathbb{Q}} S_\alpha$. Which of the following are true?
1. The Lebesgue measure of E is infinite
2. E contains a non-empty open set
3. E is path connected
4. Every open set containing E^C has infinite Legesgue measure

120. Which of the following sets are uncountable?
1. The set of all functions from $\mathbb{R}$ to {0, 1}
2. The set of all functions from $\mathbb{N}$ to {0, 1}
3. The set of all finite subsets of $\mathbb{N}$
4. The set of all subsets of $\mathbb{N}$

121. Let $A = \left\{ t \sin\left(\frac{1}{t}\right) \middle| t \in \left(0, \frac{2}{\pi}\right) \right\}$. Which of the following statements are true?
1. $\sup(A) < \frac{2}{\pi} + \frac{1}{n\pi}$ for all $n \geq 1$

2. $\inf(A) > \frac{-2}{3\pi} - \frac{1}{n\pi}$ for all $n \geq 1$
3. sup (1) = 1
4. inf (1) = -1

122. Let $C_C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous and there exists a compact set K such that f(x) = 0 for all $x \in K^C\}$. Let $g(x) = e^{-x^2}$ for all $x \in \mathbb{R}$. Which of the following statements are true?

1. There exists a sequence $\{f_n\}$ in $C_C(\mathbb{R})$ such that $f_n \to g$ uniformly
2. There exists a sequence $\{f_n\}$ in $C_C(\mathbb{R})$ such that $f_n \to g$ pointwise
3. If a sequence in $C_C(\mathbb{R})$ converges pointwise to g then it must converge uniformly to g
4. There does not exist any sequence in $C_C(\mathbb{R})$ converging pointwise to g

123. Given that

$$a(n) = \frac{1}{10^{100}} 2^n$$

$$b(n) = 10^{100} \log(n)$$

$$c(n) = \frac{1}{10^{10} n^2},$$

which of the following statements are true?
1. a(n) > c(n) for all sufficiently large n
2. b(n) > c(n) for all sufficiently large n
3. b(n) > n for all sufficiently large n
4. a(n) > b(n) for all sufficiently large n

124. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \frac{a}{1+bx^2}, \text{ a, b} \in \mathbb{R}, b \geq 0.$$

Which of the following are true?
1. f is uniformly continuous on compact intervals of $\mathbb{R}$ for all values of a and b
2. f is uniformly continuous on $\mathbb{R}$ and is bounded for all values of a and b
3. f is uniformly continuous on $\mathbb{R}$ only if b=0
4. f is uniformly continuous on $\mathbb{R}$ and unbounded if $a \neq 0$, $b \neq 0$

125. Let $\alpha = \int_0^\infty \frac{1}{1+t^2} dt.$

Which of the following are true?

1. $\frac{d\alpha}{dt} = \frac{1}{1+t^2}$
2. α is a rational number
3. log (1) = 1
4. sin (1) = 1

126. Which of the following functions are of bounded variation?
1. $x^2 + x + 1$ for $x \in (-1, 1)$
2. $\tan\left(\frac{\pi x}{2}\right)$ for $x \in (-1, 1)$
3. $\sin\left(\frac{x}{2}\right)$ for $x \in (-\pi, \pi)$
4. $\sqrt{1-x^2}$ for $x \in (-1, 1)$

127. For any $y \in \mathbb{R}$, let [y] denote the greatest integer less than or equal to y.
Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^{[y]}$. Then
1. f is continuous on $\mathbb{R}^2$
2. for every $y \in \mathbb{R}$, $x \mapsto f(x, y)$ is continuous on $\mathbb{R} \setminus \{0\}$
3. for every $x \in \mathbb{R}$, $y \mapsto f(x, y)$ is continuous on $\mathbb{R}$
4. f is continuous at no point of $\mathbb{R}^2$

128. Which of the following statements are true?
1. Every compact metric space is separable
2. If a metric space (X, d) is separable, then the metric d is not the discrete metric
3. Every separable metric space is second countable
4. Every first countable topological space is separable

129. Let X be a topological space and A be a non-empty subset of X. Then one can conclude that
1. A is dense in X, if (X\A) is nowhere dense in X
2. (X\A) is nowhere dense in X, if A is dense in X
3. A is dense in X, if the interior of (X\A) is empty
4. the interior of (X\A) is empty, if A is dense in X

December - 2018

PART – B

130. Consider the function tan x on the set

$S = \{x \in \mathbb{R}:\ x \geq 0,\ x \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{N} \cup \{0\}\}$. We say that it has a fixed point in S if $\exists\ x \in S$ such that tan x = x. Then

1. There is a unique fixed point.
2. There is no fixed point.
3. There are infinitely many fixed points.
4. There are more than one but finitely many fixed points.

131. Define $f(x) = \frac{1}{\sqrt{x}}$ for x > 0. Then f is uniformly continuous

1. on $(0,\infty)$.
2. on $[r,\infty)$ for any r>0.
3. on (0,r] for any r>0.
4. only on intervals of the form [a,b] for $0 < a < b < \infty$.

132. Consider the map f: $\mathbb{Q} \to \mathbb{R}$ defined by

(i) f(0) = 0

(ii) $f(r) = \frac{p}{10^q}$, where $r = \frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and gcd (p,q)=1. Then the map f is

1. one-to-one and onto
2. not one-to-one , but onto
3. onto but not one-to-one
4. neither one-to-one nor onto

133. Let x be a real number such that |x| < 1. Which of the following is FALSE ?

1. If $x \in \mathbb{Q}$, then $\sum_{m\geq 0} x^m \in \mathbb{Q}$
2. If $\sum_{m\geq 0} x^m \in \mathbb{Q}$, then $x \in \mathbb{Q}$
3. If $x \notin \mathbb{Q}$, then $\sum_{m\geq 0} m x^{m-1} \notin \mathbb{Q}$
4. $\sum_{m\geq 1} \frac{x^m}{m}$ converges in $\mathbb{R}$

134. Suppose that $\{x_n\}$ is a sequence of real numbers satisfying the following. For every $\in>0$, there exists n_0 such that $|x_{n+1} - x_n| < \in\ \forall\ n \geq n_0$. The sequence $\{x_n\}$ is

1. bounded but not necessarily Cauchy
2. Cauchy but not necessarily bounded
3. Convergent
4. not necessarily bounded

135. Let $A(n) = \int_n^{n+1} \frac{1}{x^3} dx$ for n ≥ 1. For $c \in \mathbb{R}$ let $\lim_{n\to\infty} n^c A(n) = L$. Then

1. L=0 if c > 3
2. L=1 if c=3
3. L=2 if c=3
4. $L=\infty$ if 0< c<3

136. Let R denote the radius of convergence of power series $\sum_{k=1}^{\infty} kx^k$. Then

1. R > 0 and the series is convergent on [-R,R]
2. R > 0 and the series converges at x = -R but does not converges at x = R
3. R>0 and the series does not converge outside (-R,R)
4. R = 0

PART – C

137. Let f : $\mathbb{R}^2 \to \mathbb{R}^2$ be a function given by f(x, y) = $(x^3 + 3xy^2 - 15x - 12y,\ x + y)$. Let S = {(x, y) $\in \mathbb{R}^2$: f is locally invertible at (x, y)}. Then

1. $S = \mathbb{R}^2 \setminus \{(0, 0)\}$
2. S is open in $\mathbb{R}^2$
3. S is dense in $\mathbb{R}^2$
4. $\mathbb{R}^2 \setminus S$ is countable

138. Let X = $\mathbb{N}$, the set of positive integers. Consider the metrices d_1, d_2 on X given by $d_1(m, n) = |m - n|$, m, n $\in$ X, $d_2(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|, m, n \in X$. Let X_1, X_2 denote the metric spaces (X, d_1), (X, d_2) respectively. Then

1. X_1 is complete
2. X_2 is complete
3. X_1 is totally bounded
4. X_2 is totally bounded

139. Let $\{u_n\}_{n\geq 1}$ be a sequence of real numbers satisfying the following conditions:

(1) $(-1)^n u_n \geq 0$, for all n ≥ 1

(2) $|u_{n+1}| < \frac{|u_n|}{2}$, for all n ≥ 13

Which of the following statements are necessarily true?

1. $\sum_{n\geq 1} u_n$ does not converge in $\mathbb{R}$.
2. $\sum_{n\geq 13} u_n$ converges to zero.
3. $\sum_{n\geq 13} u_n$ converges to a non-zero real number.
4. If $|u_{n-1}| < \frac{|u_n|}{2}$, for all $2 \leq n \leq 13$, then $\sum_{n\geq 1} u_n$ is a negative real number.

140. Let S be an infinite set. Which of the following statements are true?

1. If there is an injection from S to $\mathbb{N}$, then S is countable
2. If there is a surjection from S to $\mathbb{N}$, then S is countable
3. If there is an injection from $\mathbb{N}$ to S, then S is countable
4. If there is a surjection from $\mathbb{N}$ to S, then S is countable

141. Let p_n denote the n-th prime number, when we enumerate the prime numbers in the increasing order. For example, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on.

Let $S = \{s_n = p_{n+1} - p_n | n \in \mathbb{N}, n \geq 1\}$. Then which of the following are correct?

1. $\sup S = \infty$
2. $\limsup_{n\to\infty} s_n = \infty$
3. $\inf S < \infty$ and $\inf S = 1$
4. $\liminf_{n\to\infty} s_n \geq 2$

142. For $n \geq 1$, consider the sequence of functions

$$f_n(x) = \frac{1}{2nx+1}, g_n(x) = \frac{x}{2nx+1}$$ on the open interval (0, 1). Consider the statements:

(I) The sequence $\{f_n\}$ converges uniformly on (0, 1)
(II) The sequence $\{g_n\}$ converges uniformly on (0, 1). Then,

1. (I) is true
2. (I) is false
3. (I) is false and (II) is true
4. Both (I) and (II) are true

143. Suppose that $\{f_n\}$ is a sequence of continuous real valued functions on [0, 1] satisfying the following:

(1) $\forall x \in \mathbb{R}$, $\{f_n(x)\}$ is a decreasing sequence
(2) the sequence $\{f_n\}$ converges uniformly to 0.

Let $g_n(x) = \sum_{k=1}^{n} (-1)^k f_k(x) \forall x \in \mathbb{R}$. Then

1. $\{g_n\}$ is Cauchy with respect to the sup norm
2. $\{g_n\}$ is uniformly convergent
3. $\{g_n\}$ need not converge pointwise
4. $\exists M > 0$ such that $|g_n(x)| \leq M, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}$

144. Given $f : \left[\frac{1}{2}, 2\right] \to \mathbb{R}$, a strictly increasing function, we put $g(x) = f(x) + f(1/x)$, $x \in [1, 2]$. Consider a partition P of [1, 2] and let U (P, g) and L (P, g) denote the upper Riemann sum and lower Riemann sum of g. Then

1. for a suitable f we can have U (P, g) = L (P, g)
2. for a suitable f we can have U (P, g) ≠ L (P, g)
3. U (P, g) ≥ L (P, g) for all choices of f
4. U (P, g) < L (P, g) for all choices of f

145. Let f be a real valued continuously differentiable function of (0, 1). Set $g = f' + if$, where $i^2 = -1$ and f' is the derivative of f. Let $a, b \in (0,1)$ be two consecutive zeros of f. Which of the following statements are necessarily true?

1. If g(1) > 0, then g crosses the real line from upper half plane to lower half plane at a
2. If g(1)>0, then g crosses the real line from lower half plane to upper half plane at a
3. if g(1) g(2)≠0, then g(1), g(2) have the same sign
4. If g(1) g(2)≠0, then g(1), g(2) have opposite signs

146. Let A be an invertible real $n \times n$ matrix. Define a function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $F(x, y) = \langle Ax, y\rangle$ where $\langle x, y\rangle$ denotes the inner product of x and y. Let DF (x, y) denote the derivative of F at (x, y) which is a linear transformation from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Then

1. If $x \neq 0$, then DF (x, 0) ≠ 0
2. If $y \neq 0$, then DF (0, y) ≠ 0
3. If (x, y) ≠ (0, 0) then DF (x, y) ≠ 0
4. If x = 0 or y = 0, then DF (x, y) = 0

147. Let $X = \{(x_i)_{i\geq 1} : x_i \in \{0, 1\}$ for all $i \geq 1\}$ with the metric $d((x_i), (y_i)) = \sum_{i\geq 1}|x_i - y_i|2^{-i}$.
Let $f : X \to [0, 1]$ be the function defined by $f(x_i)_{i\geq 1} = \sum_{i\geq 1} x_i 2^{-i}$.
Choose the correct statements from below:

1. f is continuous
2. f is onto
3. f is one-to-one
4. f is open

148. Let A be a subset of $\mathbb{R}$ satisfying $A = \cap_{n\geq 1} V_n$, where for each $n \geq 1$, V_n is an open dense subset of $\mathbb{R}$. Which of the following are correct?

1. A is a non-empty set
2. A is countable
3. A is uncountable
4. A is dense in $\mathbb{R}$

149. Let $a_1 < a_2 < \ldots < a_{51}$ be given distinct natural numbers such that $1 \leq a_i \leq 100$ for all i = 1, 2, …, 51. Then which of the following are correct?

1. There exist i and j with $1 \le i < j \le 51$ satisfying a_i divides a_j.
2. There exists i with $1 \le i \le 51$ such that a_i is an odd integer
3. There exists j with $1 \le j \le 51$ such that a_j is an even integer
4. There exist i < j such that $|a_i - a_j| > 51$.

JUNE – 2019

PART – B

150. Which of the following sets is uncountable?

1. $\left\{x \in \mathbb{R} \mid \log(x) = \frac{p}{q} \text{ for some } p, q \in \mathbb{N}\right\}$
2. $\{x \in \mathbb{R} \mid (\cos(x))^n + (\sin(x))^n = 1 \text{ for some } n \in \mathbb{N}\}$
3. $\left\{x \in \mathbb{R} \mid x = \log\left(\frac{p}{q}\right) \text{ for some } p, q \in \mathbb{N}\right\}$
4. $\left\{x \in \mathbb{R} \mid \cos(x) = \frac{p}{q} \text{ for some } p, q \in \mathbb{N}\right\}$

151. Consider a sequence

$\{a_n\}$, $a_n = (-1)^n\left(\frac{1}{2} - \frac{1}{n}\right)$.

Let $b_n = \sum_{k=1}^{n} a_k \ \forall\, n \in \mathbb{N}$.

Then which of the following is true?
1. $\lim_{n\to\infty} b_n = 0$
2. $\limsup_{n\to\infty} b_n > 1/2$
3. $\liminf_{n\to\infty} b_n < -1/2$
4. $0 \le \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n \le 1/2$

152. Which of the following is true?

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge
2. $\sum_{n=1}^{\infty} \frac{1}{n}$ converges
3. $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^2}$ converges
4. $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^2}$ diverges

153. For $n \in \mathbb{N}$, which of the following is true?

1. $\sqrt{n+1} - \sqrt{n} > \frac{1}{\sqrt{n}}$ for all, except possibly finitely many n
2. $\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n}}$ for all, except possibly finitely many n
3. $\sqrt{n+1} - \sqrt{n} > 1$ for all, except possibly finitely many n
4. $\sqrt{n+1} - \sqrt{n} > 2$ for all, except possibly finitely many n

154. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and one-one function. Then which of the following is true?
1. f is onto
2. f is either strictly decreasing or strictly increasing
3. there exists $x \in \mathbb{R}$ such that $f(x) = 1$
4. f is unbounded

155. Let $g_n(x) = \frac{nx}{1+n^2x^2}, x \in [0, \infty)$. Which of the following is true as $n \to \infty$?
1. $g_n \to 0$ pointwise but not uniformly
2. $g_n \to 0$ uniformly
3. $g_n(x) \to x \quad \forall x \in [0, \infty)$
4. $g_n(x) \to \frac{x}{1+x^2} \quad \forall x \in [0, \infty)$

PART – C

156. Let $\{a_n\}_{n\ge 0}$ be a sequence of positive real numbers. Then, for $K = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, which of the following are true?

1. if $K = \infty$, then $\sum_{n=0}^{\infty} a_n r^n$ is convergent for every r > 0
2. if $K = \infty$, then $\sum_{n=0}^{\infty} a_n r^n$ is not convergent for any r > 0
3. if $K = 0$, then $\sum_{n=0}^{\infty} a_n r^n$ is convergent for every r > 0
4. if $K = 0$, then $\sum_{n=0}^{\infty} a_n r^n$ is not convergent for any r > 0

157. For $\alpha \in \mathbb{R}$, let $[\alpha]$ denote the greatest integer smaller than or equal to α. Define $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $d(x, y) = [|x - y|]$, $x, y \in \mathbb{R}$. Then which of the following are true?
1. $d(x, y) = 0$ if and only if $x = y$, $x, y \in \mathbb{R}$
2. $d(x, y) = d(y, x)$, $x, y \in \mathbb{R}$
3. $d(x, y) \le d(x, z) + d(z, y)$, $x, y, z \in \mathbb{R}$

4. d is not a metric on $\mathbb{R}$

158. Consider a function f : $\mathbb{R} \to \mathbb{R}$. Then which of the following are true?
1. f is not one-one if the graph of f intersects some line parallel to X-axis in at least two points
2. f is one-one if the graph of f intersects any line parallel to the X-axis in at most one point
3. f is surjective if the graph of f intersects every line parallel to X-axis
4. f is not surjective if the graph of f does not intersect at least one line parallel to X-axis

159. Let $f(x) = \int_1^\infty \frac{\cos t}{x^2 + t^2} dt$. Then which of the following are true?

1. f is bounded on $\mathbb{R}$
2. f is continuous on $\mathbb{R}$
3. f is not defined everywhere on $\mathbb{R}$
4. f is not continuous on $\mathbb{R}$

160. Suppose that $\{x_n\}$ is a sequence of positive reals. Let $y_n = \frac{x_n}{1 + x_n}$. Then which of the following are true?
1. $\{x_n\}$ is convergent if $\{y_n\}$ is convergent
2. $\{y_n\}$ is convergent if $\{x_n\}$ is convergent
3. $\{y_n\}$ is bounded if $\{x_n\}$ is bounded
4. $\{x_n\}$ is bounded if $\{y_n\}$ is bounded

161. Let $f(x) = \begin{cases} x\sin(1/x), & \text{for } x \in (0,1] \\ 0, & \text{for } x = 0 \end{cases}$ and g(x) = xf (x) for $0 \le x \le 1$. Then which of the following are true?
1. f is of bounded variation
2. f is not of bounded variation
3. g is of bounded variation
4. g is not of bounded variation

162. Let a < c < b, f : (a, b) $\to \mathbb{R}$ be continuous. Assume that f is differentiable at every point of (a, b) \ {c} and f′ has a limit at c. Then which of the following are true?
1. f is differentiable at c
2. f need not be differentiable at c
3. f is differentiable at c and $\lim_{x\to c}$ f′(x) = f′(3)
4. f is differentiable at c but f′(3) is not necessarily $\lim_{x\to c}$ f′(x)

163. Let F : $\mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Which of the following can be the set of discontinuities of F

1. $\mathbb{Z}$ 2. $\mathbb{N}$ 3. $\mathbb{Q}$ 4. $\mathbb{R}\backslash\mathbb{Q}$

164. Let f : $\mathbb{R}^3 \to \mathbb{R}^3$ be given by
$f(x_1, x_2, x_3) = (e^{x_2}\cos x_1, e^{x_2}\sin x_1, 2x_1 - \cos x_3)$

Consider E = {$(x_1, x_2, x_3) \in \mathbb{R}^3$: there exists an open subset U around (x_1, x_2, x_3) such that $f|_U$ is an open map}. Then which of the following are true?
1. E = $\mathbb{R}^3$
2. E is countable
3. E is not countable but not $\mathbb{R}^3$
4. $\left\{\left(x_1, x_2, \frac{\pi}{2}\right) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R}\right\}$ is a proper subset of E

165. Let X be a countable set. Then which of the following are true?
1. There exists a metric d on X such that (X, d) is complete
2. There exists a metric d on X such that (X, d) is not complete
3. There exists a metric d on X such that (X, d) is compact
4. There exists a metric d on X such that (X, d) is not compact

December - 2019

PART – B

166. Let ≤ be the usual order on the field $\mathbb{R}$ of real numbers. Define an order ≤ on $\mathbb{R}^2$ by (a, b) ≤ (c, d) if (a < c) or (a = c and b ≤ d). Consider the subset
E = $\left\{\left(\frac{1}{n}, 1 - \frac{1}{n}\right) \in \mathbb{R}^2 : n \in \mathbb{N}\right\}$. With respect to ≤ which of the following statements is true?
(1) inf(E) = (0, 1) and sup (E) = (1, 0)
(2) inf(E) does not exist but sup(E) = (1, 0)
(3) inf(E) = (0, 1) but sup(E) does not exist
(4) Both inf(E) and sup(E) do not exist.

167. Let C[0, 1] be the space of continuous real valued functions on [0, 1]. Define
$\langle f, g\rangle = \int_0^1 f(t)(g(t))^2 dt$ for all f,g∈C[0, 1]
Then which of the following statements is true?
(1) ⟨ , ⟩ is an inner product on C[0, 1]
(2) ⟨ , ⟩ is a bilinear form on C[0, 1] but is not an inner product on C[0, 1]

(3) $\langle\ ,\ \rangle$ is not a bilinear form on C[0, 1]
(4) $\langle f, f \rangle \geq 0$ for all $f \in C[0, 1]$

168. Which of the following sets is countable?

(1) The set of all functions from $\mathbb{Q}$ to $\mathbb{Q}$

(2) The set of all functions from $\mathbb{Q}$ to {0, 1}

(3) The set of all functions from $\mathbb{Q}$ to {0, 1} which vanish outside a finite set

(4) The set of all subsets of $\mathbb{N}$

169. Let E = $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. For each m $\in \mathbb{N}$ define $f_m : E \to \mathbb{R}$ by

$$f_m(x) = \begin{cases} \cos(mx)\ if\ x \geq \frac{1}{m} \\ 0\ if\ \frac{1}{m+10} < x < \frac{1}{m} \\ x\ if\ x \leq \frac{1}{m+10} \end{cases}$$

Then which of the following statements is true?
(1) No subsequence of $(f_m)_{m\geq 1}$ converges at every point of E
(2) Every subsequence of $(f_m)_{m\geq 1}$ converges at every point of E
(3) There exist infinitely many subsequences of $(f_m)_{m\geq 1}$ which converge at every point of E
(4) There exists a subsequence of $(f_m)_{m\geq 1}$ which converges to 0 at every point of E

170. Let $(x_n)_{n\geq 1}$ be a sequence of non –negative real numbers. Then which of the following is true?

(1) liminf $x_n = 0 \Rightarrow \lim x_n^2 = 0$

(2) limsup $x_n = 0 \Rightarrow \lim x_n^2 = 0$

(3) liminf $x_n = 0 \Rightarrow (x_n)_{n\geq 1}$ is bounded

(4) liminf $x_n^2 > 4 \Rightarrow$ limsup $x_n > 4$

171. Let X $\subset \mathbb{R}$ be an infinite countable bounded subset of $\mathbb{R}$. Which of the following statements is true?
(1) X cannot be compact
(2) X contains an interior point
(3) X may be closed
(4) closure of X is countable

172. What is the sum of the following series?

$$\left(\frac{1}{2.3}+\frac{1}{2^2.3}\right)+\left(\frac{1}{2^2.3^2}+\frac{1}{2^3.3^2}\right)+...+\left(\frac{1}{2^a.3^a}+\frac{1}{2^{a+1}.3^a}\right)+...$$

(1) $\frac{3}{8}$ (2) $\frac{3}{10}$ (3) $\frac{3}{14}$ (4) $\frac{3}{16}$

PART – C

173. Let $L^2([-\pi, \pi])$ be the metric space of Lebesgue square integrable functions on $[-\pi, \pi]$ with a metric d given by

$$d(f,g) = \left[\int_{-\pi}^{\pi} (f(x) - g(x))^2 dx\right]^{1/2}$$ for f, g $\in L^2([-\pi, \pi])$
Consider the subset
S = $\{\sin(2^n x) : n \in \mathbb{N}\}$ of $L^2([-\pi, \pi])$.
Which of the following statements are true?
(1) S is bounded (2) S is closed
(3) S is compact (4) S is non-compact

174. Let $f : [0, 1]^2 \to \mathbb{R}$ be a function defined by
$f(x, y) = \frac{xy}{x^2 + y^2}$ if either $x \neq 0$ or $y \neq 0$ = 0 if x = y = 0.
Then which of the following statements are true?
(1) f is continuous at (0, 0)
(2) f is a bounded function
(3) $\int_0^1 \int_0^1 f(x, y)\, dxdy$ exists
(4) f is continuous at (1, 0)

175. Let p(x) be a polynomial function in one variable of odd degree and g be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then which of the following statements are true.

(1) $\exists$ a point $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$
(2) If g is a polynomial function then there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$
(3) If g is a bounded function there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$
(4) There is a unique point $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$

176. Let f(x) be a real polynomial of degree 4. Suppose f(-1)=0, f(0)=0, f(1)=1 and $f^{(1)}(0)$ =0, where $f^{(k)}(1)$ is the value of k^{th} derivative of f(x) at x=a. Which of the following statements are true?
(1) There exists a$\in$(-1,1) such that $f^{(3)}(1) \geq 3$
(2) $f^{(3)}(1) \geq 3$ for all a $\in$ (-1, 1)
(3) $0 < f^{(3)}(0) \leq 2$
(4) $f^{(3)}(0) \geq 3$

177. Let (X, d) be a compact metric space. Let T : X → X be a continuous function satisfying

$\inf_{n\in\mathbb{N}} d(T^n(x), T^n(y)) \neq 0$ for every x, y ∈ X with x ≠ y. Then which of the following statements are true?

(1) T is a one-one function
(2) T is not a one-one function
(3) Image of T is closed in X
(4) If X is finite, then T is onto

178. For each natural number n ≥ 1, let $a_n = \frac{n}{10^{[\log_{10} n]}}$, where [x] = smallest integer greater than or equal to x. Which of the following statements are true?

(1) $\liminf_{n\to\infty} a_n = 0$
(2) $\liminf_{n\to\infty} a_n$ does not exist
(3) $\liminf_{n\to\infty} a_n = 0.15$
(4) $\limsup_{n\to\infty} a_n = 1$

179. Let $U \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a C^∞ - function. Suppose that for every x ∈ U, the derivative at x, df_x, is non-singular. Then which of the following statements are true?

(1) If V ⊂ U is open then f(V) is open in $\mathbb{R}^n$
(2) f : U → f(U) is a homomorphism
(3) f is one-one
(4) If V ⊂ U is closed, then f(V) is closed in $\mathbb{R}^n$.

180. Let n be a fixed natural number. Then the series $\sum_{m\geq n} \frac{(-1)^m}{m}$ is

(1) Absolutely convergent
(2) Divergent
(3) Absolutely convergent if n > 100
(4) Convergent

181. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real numbers. Then

(1) Every subsequence of $\{a_n\}_{n\geq 1}$ is convergent
(2) There is exactly one subsequence of $\{a_n\}_{n\geq 1}$ which is convergent
(3) There are infinitely many subsequences of $\{a_n\}_{n\geq 1}$ which are convergent
(4) There is a subsequence of $\{a_n\}_{n\geq 1}$ which is convergent

182. Let N ≥ 5 be an integer. Then which of the following statements are true?

(1) $\sum_{n=1}^{N} \frac{1}{n} \leq 1 + \log N$
(2) $\sum_{n=1}^{N} \frac{1}{n} < 1 + \log N$
(3) $\sum_{n=1}^{N} \frac{1}{n} \leq \log N$
(4) $\sum_{n=1}^{N} \frac{1}{n} \geq \log N$

183. Let f : [0, 1] → ℝ be a monotonic function with $f\left(\frac{1}{4}\right) f\left(\frac{3}{4}\right) < 0.$

Suppose sup{x ∈ [0, 1] : f(x) < 0} = α. Which of the following statements are correct?

(1) f(1) < 0
(2) If f is increasing, then f(1) ≤ 0
(3) If f is continuous and $\frac{1}{4} < \alpha < \frac{3}{4}$, then f(1) = 0
(4) If f is decreasing, then f(1) < 0

184. Which of the following statements are true?

(1) There exist three mutually disjoint subsets of ℝ, each of which is countable and dense in ℝ
(2) For each n ∈ ℕ, there exist n mutually disjoint subsets of ℝ each of which is countable and dense in ℝ
(3) There exist countably infinite number of mutually disjoint subsets of ℝ, each of which is countable and dense in ℝ
(4) There exist uncountable number of mutually disjoint subsets of ℝ, each of which is countable and dense in ℝ

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<u>PART – B</u>

185. Let $\{E_n\}$ be a sequence of subsets of ℝ. Define

$$\limsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

$$\liminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

Which of the following statements is true?

(1) $\limsup_n E_n = \liminf_n E_n$

(2) $\limsup_n E_n = \{x : x \in E_n \text{ for some } n\}$

(3) $\liminf_n E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$

(4) $\liminf_n E_n = \{x : x \in E_n \text{ for infinitely many } n\}$

186. f : ℕ → ℕ be a bounded function. Which of the following statements is NOT true?

(1) $\limsup_{n\to\infty} f(n) \in \mathbb{N}$

(2) $\liminf_{n\to\infty} f(n) \in \mathbb{N}$

(3) $\liminf_{n\to\infty} (f(n)+n) \in \mathbb{N}$

(4) $\limsup_{n\to\infty} (f(n)+n) \notin \mathbb{N}$

187. Which of the following statements is true?

(1) There are at most countably many continuous maps from $\mathbb{R}^2$ to ℝ.

(2) There are at most finitely many continuous surjective maps from $\mathbb{R}^2$ to ℝ.

(3) There are infinitely many continuous injective maps from $\mathbb{R}^2$ to ℝ.

(4) There are no continuous bijective maps from $\mathbb{R}^2$ to ℝ.

188. The series

$\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^{\log_e n}}, x \in \mathbb{R}$ converges

(1) only for x = 0

(2) uniformly only for x ∈ [-π, π]

(3) uniformly only for x ∈ ℝ\{nπ : n ∈ ℤ}

(4) uniformly for all x ∈ ℝ

189. Given $(a_n)_{n\geq 1}$ a sequence of real numbers, which of the following statements is true?

(1) $\sum_{n\geq 1} (-1)^n \frac{a_n}{1+|a_n|}$ converges

(2) There is a subsequence $(a_{n_k})_{k\geq 1}$ such that $\sum_{k\geq 1} \frac{a_{n_k}}{1+|a_{n_k}|}$ converges

(3) There is a number b such that $\sum_{n\geq 1} \left| b - \frac{a_n}{1+|a_n|} \right| (-1)^n$ converges

(4) There is a number b and a subsequence $(a_{n_k})_{k\geq 1}$ such that $\sum_{k\geq 1} \left| b - \frac{a_{n_k}}{1+|a_{n_k}|} \right|$ converges

190. Given f, g are continuous functions on [0, 1] such that f(0) = f(1) = 0; g(0) = g(1) = 1 and f(1/2) > g(1/2). Which of the following statements is true?

(1) There is no t∈[0, 1] such that f(t) = g(t)

(2) There is exactly one t∈[0, 1] such that f(t) = g(t)

(3) There are at least two t∈[0, 1] such that f(t) = g(t)

(4) There are always infintely many t∈[0, 1] such that f(t) = g(t)

PART – C

191. Which of the following sets are in bijection with ℝ?

(1) Set of all maps from {0, 1} to ℕ

(2) Set of all maps from ℕ to {0, 1}

(3) Set of all subsets of ℕ

(4) Set of all susets of ℝ

192. Which of the following statements are true?

(1) The series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$ is convergent

(2) The series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n+n}}$ is absolutely convergent

(3) The series $\sum_{n\geq 1} \frac{[1+(-1)^n]\sqrt{n}+\log n}{n^{3/2}}$ is convergent

(4) The series $\sum_{n\geq 1} \frac{((-1)^n\sqrt{n}+1)}{n^{3/2}}$ is convergent

193. Let f : $\mathbb{R}^2$ → ℝ be defined by

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Define $g(x,y) = \sum_{n=1}^{\infty} \frac{f((x-n),(y-n))}{2^n}$

Which of the following statements are true?

(1) The function h(y) = g(c, y) is continuous on ℝ for all c

(2) g is continuous from $\mathbb{R}^2$ into $\mathbb{R}$
(3) g is not a well-defined function
(4) g is continuous on $\mathbb{R}^2 \setminus \{(k, k)\}_{k\in\mathbb{N}}$

194. Consider the series

$$A(x) = \sum_{n=0}^{\infty} x^n (1-x) \text{ and}$$

$$B(x) = \sum_{n=0}^{\infty} (-1)^n x^n (1-x)$$

where $x \in [0, 1]$.
Which of the following statements are true?
(1) Both A(x) and B(x) converge pointwise
(2) Both A(x) and B(x) converge uniformly
(3) A(x) converges uniformly but B(x) does not
(4) B(x) converges uniformly but A(x) does not

195. For $p \in \mathbb{R}$, consider the improper integral

$$I_p = \int_0^1 t^p \sin t \, dt.$$

Which of the following statements are true?
(1) I_p is convergent for p = -1/2
(2) I_p is divergent for p = -3/2
(3) I_p is convergent for p = 4/3
(4) I_p is divergent for p = -4/3

196. Suppose that $\{f_n\}$ is a sequence of real-valued functions on $\mathbb{R}$. Suppose it converges to a continuous function f uniformly on each closed and bounded subset of $\mathbb{R}$. Which of the following statements are true?
(1) The sequence $\{f_n\}$ converges to f uniformly on $\mathbb{R}$
(2) The sequence $\{f_n\}$ converges to f pointwise on $\mathbb{R}$
(3) For all sufficiently large n, the function f_n is bounded
(4) For all sufficiently large n the function f_n is continuous

197. Let $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$. Which of the following statements are true?
(1) Both f and g are uniformly continuous on $\mathbb{R}$
(2) f is uniformly continuous on every interval of the form $[a, +\infty)$, $a \in \mathbb{R}$
(3) g is uniformly continuous on $\mathbb{R}$
(4) f(x) g(x) is uniformly continuous on $\mathbb{R}$

198. Define

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

Which of the following statements are true?
(1) f is discontinuous at (0, 0)
(2) f is continuous at (0, 0)
(3) all directional derivatives of f at (0, 0) exist
(4) f is not differentiable at (0, 0)

199. Define

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}.$$

Which of the following statements are true?
(1) f is continuous at (0, 0)
(2) f is bounded in a neighbourhood of (0, 0)
(3) f is not bounded in any neighbourhood of (0, 0)
(4) f has all directional derivatives at (0, 0)

200. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$p(x,y) = \begin{cases} |x| & \text{if } x \neq 0 \\ |y| & \text{if } x = 0 \end{cases}$$

Which of the following statements are true?
(1) p(x, y) = 0 if and only if x = y = 0
(2) $p(x, y) \geq 0$ for all x, y
(3) $p(\alpha x, \alpha y) = |\alpha|\, p(x, y)$ for all $\alpha \in \mathbb{R}$ and for all x, y
(4) $p(x_1 + x_2, y_1 + y_2) \leq p(x_1, y_1) + p(x_2, y_2)$ for all (x_1, y_1), (x_2, y_2)

201. Consider the subset of $\mathbb{R}^2$ defined as follows:
$A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x - 1)(x - 2)(y - 3)(y + 4) = 0\}$
Which of the following statements are true?
(1) A is connected (2) A is compact
(3) A is closed (4) A is dense

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<u>PART – B</u>

202. Let (X, d) be a metric space and let $f : X \to X$ be a function such that $d(f(x), f(y)) \leq d(x,$

y) for every x, y ∈ X. Which of the following statements is necessarily true?
(1) f is continuous
(2) f is injective
(3) f is surjective
(4) f is injective and if and only if f is surjective

203. Let S = {n: 1 ≤ n ≤ 999; 3|n or 37|n}. How many integers are there in the set S^c = {n: 1 ≤ n ≤ 999; n ∉ S}?
(1) 639 (2) 648
(3) 666 (4) 990

204. Let f, g: $\mathbb{R} \mapsto \mathbb{R}$ be given by and f(x) = x^2 and g(x) = sinx
Which of the following functions is uniformly continuous on $\mathbb{R}$?
(1) h(x) = g(f(x)) (2) h(x) = g(x) f(x)
(3) h(x) = f(g(x)) (4) h(x) = f(x) + g(x)

205. Let S = {1, 2, …, 100} and let A = {1, 2, …, 10} and B = {41, 42, …, 50}. What is the total number of subsets of S, which have non-empty intersection with both A and B?
(1) $\frac{2^{100}}{2^{20}}$ (2) $\frac{100!}{10!10!}$
(3) $2^{80}(2^{10}-1)^2$ (4) $2^{100} - 2(2^{10})$

206. Consider the sequence $\{a_n\}_{n\geq 1}$, where

$$a_n = 3 + 5\left(-\frac{1}{2}\right)^n + (-1)^n\left(\frac{1}{4} + (-1)^n \frac{2}{n}\right)$$

Then the interval $\left(\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} a_n\right)$ is given by
(1) (-2, 8) (2) $\left(\frac{11}{4}, \frac{13}{4}\right)$
(3) (3, 5) (4) $\left(\frac{1}{4}, \frac{7}{4}\right)$

207. Let $S_1 = \frac{1}{3} - \frac{1}{2}\times\frac{1}{3^2} + \frac{1}{3}\times\frac{1}{3^3} - \frac{1}{4}\times\frac{1}{3^4} + \ldots$
and
$$S_2 = \frac{1}{4} + \frac{1}{2}\times\frac{1}{4^2} + \frac{1}{3}\times\frac{1}{4^3} + \frac{1}{4}\times\frac{1}{4^4} + \ldots$$
Which of the following identities is true?
(1) $3S_1 = 4S_2$ (2) $4S_1 = 3S_2$
(3) $S_1 + S_2 = 0$ (4) $S_1 = S_2$

208. $\lim_{n\to\infty} \frac{1}{n}(1 + \sqrt{2} + \sqrt[3]{3} + \ldots + \sqrt[n]{n})$
(1) is equal to 0 (2) is equal to 1
(3) is equal to 2 (4) does not exist

PART – C

209. For non-negative integers k ≥ 1 define

$$f_k(x) = \frac{x^k}{(1+x)^2}\ \forall\, x \geq 0$$

Which of the following statements are true?
(1) For each k, f_k is a function of bounded variation on compact intervals
(2) For every k, $\int_0^\infty f_k(x)\,dx < \infty$
(3) $\lim_{k\to\infty} \int_n^1 f_k(x)\,dx$ exists
(4) The sequence of functions f_k converge uniformly on [0, 1] as k → ∞

210. In which of the following cases does there exist a continuous and onto function f : X → Y?
(1) X = (0, 1), Y = (0, 1]
(2) X = [0, 1], Y = (0, 1]
(3) X = (0, 1), Y = $\mathbb{R}$
(4) X = (0, 2), Y = {0, 1}

211. Let A ⊆ $\mathbb{R}$ and let f : $\mathbb{R} \to \mathbb{R}$ be continuous. Which of the following statements are true?
(1) If A is closed then f(1) is closed
(2) If A is bounded then $f^{-1}(1)$ is bounded
(3) If A is closed and bounded then f(1) is closed and bounded
(4) If A is bounded then f(1) is bounded

212. Let f : $\mathbb{R}^2 \to \mathbb{R}$ be a bounded function such that for each t ∈ $\mathbb{R}$, the functions g_t and h_t given by $g_t(y)$ = f(t, y) and $h_t(x)$ = f(x, t) are non decreasing functions. Which of the following statements are necessarily true?
(1) k(x) = f(x, x) is a non-decreasing function
(2) Number of discontinuities of f is at most countably infinite
(3) $\lim_{(x,y)\to(+\infty, +\infty)}$ f(x,y) exists
(4) $\lim_{(x,y)\to(+\infty, -\infty)}$ f(x,y) exists

213. Let f : $\mathbb{R}^3 \to \mathbb{R}^2$ be a C^1 function with f(0, 0, 0) = (0, 0). Let A denote the derivative of f at (0, 0, 0). Let g : $\mathbb{R}^3 \to \mathbb{R}$ be the function given by g(x, y, z) = xy + yz + zx + x + y + z.

Let h : $\mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by h = (f, g).

In which of the following cases, will the function h admit a differentiable inverse in some open neighbourhood of (0, 0, 0)?

(1) $A=\begin{pmatrix}1 & 2 & 1\\ 0 & 0 & 0\end{pmatrix}$

(2) $A=\begin{pmatrix}2 & 2 & 2\\ 6 & 5 & 2\end{pmatrix}$

(3) $A=\begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$

(4) $A=\begin{pmatrix}4 & 2 & 4\\ 0 & 3 & 2\end{pmatrix}$

214. Consider A = {1, 1/2, 1/3, …, 1/n, …, |n ∈ $\mathbb{N}$} and B = A ∪ {0}.

Both the sets are endowed with subspace topology from $\mathbb{R}$. Which of the following statements are true?

(1) A is a closed subset of $\mathbb{R}$

(2) B is a closed subset of $\mathbb{R}$

(3) A is homeomorphic to $\mathbb{Z}$, where $\mathbb{Z}$ has subspace topology from $\mathbb{R}$

(4) B is homeomorphic to $\mathbb{Z}$, where $\mathbb{Z}$ has subspace topology from $\mathbb{R}$

215. Which of the following statements are true about subsets of $\mathbb{R}^2$ with the usual topology?

(1) A is connected if and only if its closure $\overline{A}$ is connected

(2) Intersection of two connected subsets is connected

(3) Union of two compact subsets is compact

(4) There are exactly two continuous functions from $\mathbb{Q}^2$ to the set {(0, 0), (1, 1)}

216. Let f : [0, 1] → $\mathbb{R}$ be a continuous function such that $\int_0^t f(x)\,dx = \int_t^1 f(x)\,dx,$ for every t ∈ [0, 1]. Then which of the following are necessarily true?

(1) f is differentiable on (0, 1)

(2) f is monotonic on [0, 1]

(3) $\int_0^1 f(x)\,dx = 1$

(4) f(x) > 0 for all rationals x ∈ [0, 1]

217. Let $\mathbb{R}^+$ denote the set of all positive real numbers. Suppose that f : $\mathbb{R}^+ \mapsto \mathbb{R}$ is a differentiable function. Consider the function g(x) = $e^x f(x)$. Which of the following are true?

(1) If $\lim_{x\to\infty} f(x) = 0$ then $\lim_{x\to\infty} f'(x) = 0$

(2) If $\lim_{x\to\infty}(f(x) + f'(x)) = 0$ then
$$\lim_{\substack{x\to\infty\\ y\to\infty}} \frac{g(x)-g(y)}{e^x - e^y} = 0$$

(3) If $\lim_{x\to\infty} f'(x) = 0$ then $\lim_{x\to\infty} f(x) = 0$

(4) If $\lim_{x\to\infty}(f(x) + f'(x)) = 0$ then $\lim_{x\to\infty} f(x) = 0$

218. Let (a_n) and (b_n) be two sequences of real numbers and E and F be two subsets of $\mathbb{R}$. Let E + F = {a + b: a ∈ E, b ∈ F}. Assume that the right hand side is well defined in each of the following statements. Which of the following statements are true?

(1) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$

(2) $\limsup(E+F) \le \limsup E + \limsup F$

(3) $\liminf_{n\to\infty}(a_n + b_n) \le \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$

(4) $\liminf\,(E+F) = \liminf\, E + \limsup F$

219. Let X be a topological space and E be a subset of X. Which of the following statements are correct?

(1) E is connected implies $\overline{E}$ is connected

(2) E is connected implies ∂E is connected

(3) E is connected implies $\overline{E}$ is path connected

(4) E is compact implies $\overline{E}$ is compact.

220. Let Y be a nonempty bounded, open subset of $\mathbb{R}^n$ and let $\overline{Y}$ denote its closure. Let $\{U_j\}_{j\ge1}$ be a collection of open sets in $\mathbb{R}^n$ such that $\overline{Y} \subseteq \bigcup_{j\ge1} U_j$. Which of the following statements are true?

(1) There exist finitely many positive integers $j_1, \ldots, j_N$ such that $Y \subseteq \bigcup_{k=1}^{N} U_{jk}$

(2) There exists a positive integer N such that $Y \subseteq \bigcup_{j=1}^{N} U_j$

(3) For every subsequence $j_1, j_2, \ldots$ we have $Y \subseteq \bigcup_{k=1}^{\infty} U_{jk}$

(4) There exists a subsequence $j_1, j_2, \ldots$ such that $Y = \bigcup_{k=1}^{\infty} U_{jk}$

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PART – B

221. Let $a_n = n + n^{-1}$. Which of the following is true for the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a_{n+1}}{n!}$?
(1) It does not converge
(2) It converges to $e^{-1} - 1$
(3) It converges to e^{-1}
(4) It converges to $e^{-1} + 1$

222. Consider the series $\sum_{n=3}^{\infty}\frac{a^n}{n^b(\log_e n)^c}$. For which values of a, b, c ∈ ℝ, does the series NOT converge?
(1) $|a| < 1$, b, c ∈ ℝ
(2) a = 1, b > 1, c ∈ ℝ
(3) a = 1, 1 ≥ b, c < 1
(4) a = -1, b ≥ 0, c > 0

223. Suppose $(a_n)_{n\geq1}$ and $(b_n)_{n\geq1}$ are two bold sequences of real numbers. Which of the following is true?
(1)
$\limsup_{n\to\infty}(a_n + (-1)^n b_n) = \limsup_{n\to\infty} a_n + \left|\limsup_{n\to\infty} b_n\right|$
(2)
$\limsup_{n\to\infty}(a_n + (-1)^n b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
(3)
$\limsup_{n\to\infty}(a_n + (-1)^n b_n) \leq \limsup_{n\to\infty} a_n + \left|\limsup_{n\to\infty} b_n\right| + \left|\liminf_{n\to\infty} b_n\right|$
(4) $\limsup_{n\to\infty}(a_n + (-1)^n b_n)$ may not exist

224. Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(t) = (n + 2)(n + 1)t^n(1 - t)$, for all t in [0, 1]. Which of the following is true?
(1) The sequence (f_n) converges uniformly
(2) The sequence (f_n) converges pointwise but not uniformly
(3) The sequence (f_n) diverges on [0, 1)
(4) $\lim_{n\to\infty}\int_0^1 f_n(t)\,dt = \int_0^1 \lim_{n\to\infty} f_n(t)\,dt$

225. Let X, Y be defined by $X = \{(x_n)_{n\geq1} : \limsup_{n\to\infty} x_n = 1 \; where \; x_n \in \{0,1\}\}$ and $Y = \{(x_n)_{n\geq1} : \lim_{n\to\infty} x_n$ does not exists where $x_n \in \{0, 1\}\}$. Which of following is true
(1) X, Y are countable
(2) X is countable and Y is uncountable
(3) X is uncountable and Y is countable
(4) X, Y are uncountable

226. Let us define a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers to be a Fibonacci-like sequence if $a_n = a_{n-1} + a_{n-2}$ for n ≥ 3. What is the dimension of the ℝ vector space of Fibonacci-like sequences?
(1) 1
(2) 2
(3) infinite and countable
(4) infinite and uncountable

227. Let D denote a proper dense subset of a metric space X. Suppose that f : D → ℝ is a uniformly continuous function. For p ∈ X, let $B_n(p)$ denote the set
$\left\{x \in D : d(x, p) < \frac{1}{n}\right\}$.
Consider $W_p = \bigcap_n \overline{f(B_n(p))}$. Which of the following statements is true?
(1) W_p may be empty for some p in X.
(2) W_p is not empty for every p in X and is contained in f(4).
(3) W_p is a singleton for every p.
(4) W_p is empty for some p and singleton for some p.

228. Let X be a connected metric space with atleast two points. Which of the following is necessarily true?
(1) X has finitely many points
(2) X has countably many points but is not finite
(3) X has uncountably many points
(4) No such X exists

PART – C

229. Consider the following assertions:
S1: $e^{\cos(t)} \neq e^{2022\sin(t)}$ for all t ∈ (0, π).
S2: For each x > 0, there exists a t ∈ (0, x) such that $x = \log_e(1 + xe^t)$.
S3: $e^{|\sin(x)|} \leq e^{|x|}$ for all x ∈ (-1, 1).
Which of the above assertions are correct?
(1) Only S1
(2) Only S3
(3) Only S1 and S2
(4) Only S2 and S3

230. Let $\Omega = \bigcup_{i=1}^{5}(i, i+1) \subset \mathbb{R}$ and f : Ω → ℝ be a differentiable function such that f′(x) =

0 for all x ∈ Ω and let g : ℝ → ℝ be any function. Which of the following statements are true?

(1) If g is continuous, then (g ∘ f) (Ω) is a compact set in ℝ.
(2) If g is differentiable and g′(x) > 0 for all x ∈ ℝ, then (g ∘ f) (Ω) has precisely 5 elements.
(3) If g is continuous and surjective, then (g ∘ f) (Ω) ∩ ℚ ≠ ∅.
(4) If g is differentiable, then {e^x : x ∈ (g ∘ f) (Ω)} does not contain any non-empty open interval.

231. Let [x] denote the integer part of x for any real number x. Which of the following sets have non-zero Lebesgue measure?
(1) {x ∈ [1, ∞) : $\lim_{n\to\infty}[x]^n$ exists}
(2) {x ∈ [1, ∞) : $\lim_{n\to\infty}[x^n]$ exists}
(3) {x ∈ [1, ∞) : $\lim_{n\to\infty}n[x]^n$ exists}
(4) {x ∈ [1, ∞) : $\lim_{n\to\infty}[1-x]^n$ exists

232. Let (X, d) be a finite non-singleton metric space. Which of the following statements are true?
(1) There exists A ⊆ X such that A is not open in X.
(2) X is compact.
(3) X is not connected.
(4) There exists a function f : X → ℝ such that f is not continuous.

233. What is the largest positive real number δ such that whenever |x – y| < δ, we have |cosx – cosy| < $\sqrt{2}$?

(1) $\sqrt{2}$ (2) $\frac{3}{2}$

(3) $\frac{\pi}{2}$ (4) 2

234. Let a, b ∈ ℝ such that a < b and let f : (a, b) → ℝ be a continuous function. Which of the following statements are true?
(1) If f is uniformly continuous then there exist α ≥ 0 and β ≥ 0 satisfying |f(x) – f(y)| ≤ α|x – y| + β, for all x, y in (a, b).
(2) For every c, d such that [c, d] ⊆ (a, b), if f restricted to [c, d] is uniformly continuous then f is uniformly continuous.
(3) If f is strictly increasing and bounded than f is uniformly continuous.
(4) If f is uniformly continuous then it maps Cauchy sequences into convergent sequences.

235. Consider the function f : $\mathbb{R}^2$ → ℝ defined by

$$f(x,y)=\begin{cases}(x-y)^2\sin\frac{1}{(x-y)} & if \quad x\neq y\\ 0 & if \quad x=y\end{cases}$$

Which of following statements are true
(1) f is continuous at (0, 0)
(2) The partial derivative f_x does not exist at (0, 0)
(3) The partial derivative f_x is continuous at (0, 0)
(4) f is differentiable at (0, 0)

236. Which of the given sequences (a_n) satisfy following identity?

$$\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty} a_n$$

(1) $a_n = \frac{1}{n}$ for all n

(2) $a_n = (-1)^n\left(n+\frac{1}{n}\right)$ for all n

(3) $a_n = 1+\frac{(-1)^n}{n}$ for all n

(4) (a_n) is an enumeration of all rational numbers in (-1, 1)

237. For α ≥ 0, define $a_n = \frac{1+2^\alpha+...+n^\alpha}{n^{\alpha+1}}$.

What is the value of $\lim_{n\to\infty} a_n$?

(1) The limit does not exist

(2) $\frac{1}{\alpha^2+1}$

(3) $\frac{1}{\alpha+1}$

(4) $\frac{1}{\alpha^2+\alpha+1}$

238. Consider the function f : $\mathbb{R}^2$ → ℝ defined by $f(x,y)=x^{\frac{1}{3}}y^{\frac{1}{3}}$ (x, y ∈ ℝ).
Which of following statements are true?
(1) The directional derivative of f exists at (0, 0) in some direction
(2) The partial derivative f_x does not exist at (0, 0)
(3) f is continuous at (0, 0)

(4) f is not differentiable at (0, 0)

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PART – B

239. Consider ℝ with the usual topology. Which of the following assertions is correct?
(1) A finite set containing 33 elements has atleast 3 different Hausdorff topologies.
(2) Let X be a non-empty finite set with a Hausdorff topology. Consider X × X with the product topology. Then, every function f : X × X → ℝ is continuous.
(3) Let X be a discrete topological space having infinitely many elements. Let f : ℝ → X be a continuous function and g : X → ℝ be any non-constant function. Then the range of g ∘ f contains at least 2 elements.
(4) If a non-empty metric space X has a finite dense subset, then there exists a discontinuous function f : X → ℝ.

240. How many real roots does the polynomial $x^3 + 3x - 2023$ have?
(1) 0 (2) 1
(3) 2 (4) 3

241. Suppose S is an infinite set. Assuming that the axiom of choice holds, which of the following is true?
(1) S is in bijection with the set of rational numbers.
(2) S in in bijection with the set of real numbers.
(3) S is in bijection with S × S.
(4) S is in bijection with the power set of S.

242. Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = (-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$. Which of the following statements is true?
(1) The series is divergent.
(2) The series is convergent.
(3) The series is conditionally convergent.
(4) The series is absolutely convergent.

243. Let x, y ∈ [0, 1] be such that x ≠ y. Which of the following statements is true for every ∈ > 0?
(1) There exists a positive integer N such that $|x - y| < 2^n \in$ for every integer.
(2) There exists a positive integer N such that $2^n \in < |x - y|$ for every integer.
(3) There exists a positive integer N such that $|x - y| < 2^{-n} \in$ for every integer.
(4) For every positive integer N, $|x - y| < 2^{-n} \in$ for some integer n ≥ N.

244. Which one of the following functions is uniformly continuous on the interval (0, 1)?
(1) $f(x) = \sin\frac{1}{x}$
(2) $f(x) = e^{-1/x^2}$
(3) $f(x) = e^x \cos\frac{1}{x}$
(4) $f(x) = \cos x \cos\frac{\pi}{x}$

245. Which of the following assertions is correct?
(1) $\lim_n \sup e^{\cos\left(\frac{n\pi+(-1)^n 2e}{2n}\right)} > 1.$
(2) $\lim_n e^{\log_e\left(\frac{n\pi^2+(-1)^n e^2}{7n}\right)}$ does not exist.
(3) $\lim_n \inf e^{\sin\left(\frac{n\pi+(-1)^n 2e}{2n}\right)} < \pi.$
(4) $\lim_n e^{\tan\left(\frac{n\pi^2+(-1)^n e^2}{7n}\right)}$ does not exist.

PART – C

246. Consider the following two sequences $\{a_n\}$ and $\{b_n\}$ given by

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n},$$

$$b_n = \frac{1}{n}.$$

Which of the following statements are true?
(1) $\{a_n\}$ converges to log 2 and has the same convergence rate as the sequence $\{b_n\}$.
(2) $\{a_n\}$ converges to log 4 and has the same convergence rate as the sequence $\{b_n\}$.
(3) $\{a_n\}$ converges to log 2, but does not have the same convergence rate as the sequence $\{b_n\}$.

(4) $\{a_n\}$ does not converge.

247. Which of the following statements are correct?
(1) The set of open right half-planes is a basis for the usual (Euclidean) topology on $\mathbb{R}^2$.
(2) The set of lines parallel to Y-axis is a basis for the dictionary order topology on $\mathbb{R}^2$.
(3) The set of open rectangles is a basis for the usual (Euclidean) topology on $\mathbb{R}^2$.
(4) The set of line segments (without end points) parallel to Y-axis is a basis for the dictionary order topology on $\mathbb{R}^2$.

248. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 - y^3$.
Which of the following statements are true?
(1) There is no continuous real-valued function g defined on any interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.
(2) There is exactly one continuous real-valued function g defined on an interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.
(3) There is exactly one differentiable real-valued function g defined on an interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.
(4) There are two distinct differentiable real-valued functions g on an interval of $\mathbb{R}$ containing 0 such that $f(x, g(x)) = 0$.

249. Let $X = \prod_{n=1}^{\infty}[0,1]$, that is, the space of sequences $\{x_n\}_{n\geq 1}$ with $x_n \in [0, 1]$, $n \geq 1$. Define the metric $d : X \times X \to [0, \infty)$ by $d(\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}) = \sup_{n\geq 1} \frac{|x_n - y_n|}{2^n}$
Which of the following statements are true?
(1) The metric topology on X is finer than the product topology on X.
(2) The metric topology on X is coarser than the product topology on X.
(3) The metric topology on X is same as the product topology on X.
(4) The metric topology on X and the product topology on X are not comparable.

250. Which of the following are true?
(1) For $n \geq 1$, the sequence of functions $f_n : (0, 1) \to (0, 1)$ defined by $f_n(x) = x^n$ is uniformly convergent.
(2) For $n \geq 1$, the sequence of functions $f_n: (0, 1) \to (0, 1)$ defined by $f_n(x) = \frac{x^n}{\log(n+1)}$ is uniformly convergent.
(3) For $n \geq 1$, the sequence of functions $f_n: (0, 1) \to (0, 1)$ defined by $f_n(x) = \frac{x^n}{1+x^n}$ is uniformly convergent.
(4) For $n \geq 1$, the sequence of functions $f_n: (0, 1) \to (0, 1)$ defined by $f_n(x) = \frac{x^n}{1+nx^n}$ is not uniformly convergent.

251. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \frac{1}{4} + x - x^2$. Given $a \in \mathbb{R}$, let us define the sequence $\{x_n\}$ by $x_0 = a$ and $x_n = f(x_{n-1})$ for $n \geq 1$.
Which of the following statements are true?
(1) If a = 0, then the sequence $\{x_n\}$ converges to $\frac{1}{2}$.
(2) If a = 0, then the sequence $\{x_n\}$ converges to $-\frac{1}{2}$.
(3) The sequence $\{x_n\}$ converges for every $a \in \left(-\frac{1}{2}, \frac{3}{2}\right)$, and it converges to $\frac{1}{2}$.
(4) If a = 0, then the sequence $\{x_n\}$ does not converge.

252. Define $f : \mathbb{R}^4 \to \mathbb{R}$ by $f(x, y, z, w) = xw - yz$.
Which of the following statements are true?
(1) f is continuous
(2) if $U = \{(x, y, z, w) \in \mathbb{R}^4: xy + zw = 0, x^2 + z^2 = 1, y^2 + w^2 = 1\}$, then f is uniformly continuous on U.
(3) if $V = \{(x, y, z, w) \in \mathbb{R}^4 : x = y, z = w\}$, then f is uniformly continuous on V.
(4) if $W = \{(x, y, z, w) \in \mathbb{R}^4: 0 \leq x + y + z + w \leq 1\}$, then f is unbounded on W.

253. Let μ denote the Lebesgue measure on ℝ and μ* be the associated Lebesgue outer measure. Let A be a non-empty subset of [0, 1]. Which of the following statements are true?
(1) If both interior and closure of A have the same outer measure, then A is Lebesgue measurable.
(2) If A is open, then A is Lebesgue measureable and μ(A) > 0.
(3) If A is not Lebesgue measurable, then the set of irrationals in A must have positive outer measure.
(4) If μ*(A) = 0, then A has empty interior.

254. Define a function f : ℝ → ℝ by

$$f(x)=\begin{cases}\sin(\pi/x) & when\, x\neq 0\\ 0 & when\, x=0.\end{cases}$$

On which of the following subsets of ℝ, the restriction of f is a continuous function?
(1) [-1, 1]
(2) (0, 1)
(3) {0} ∪ {(1/n): n ∈ ℕ}
(4) $\{1/2^n : n \in \mathbb{N}\}$

255. Let $\{x_n\}$ be a sequence of positive real numbers. If $\sigma_n=\frac{1}{n}(x_1+x_2+...x_n)$, then which of the following are true? (Here lim sup denotes the limit supremum of a sequence.)
(1) If lim sup$\{x_n\}$ = ℓ and $\{x_n\}$ is decreasing, then lim sup$\{\sigma_n\}$ = ℓ.
(2) lim sup$\{x_n\}$ = ℓ if and only if lim sup $\{\sigma_n\}$ = ℓ.
(3) If $\limsup\left\{n\left(\frac{x_n}{x_{(n+1)}}-1\right)\right\}<1$, then $\Sigma_n x_n$ is convergent.
(4) If $\limsup\left\{n\left(\frac{x_n}{x_{(n+1)}}-1\right)\right\}<1$, then $\Sigma_n x_n$ is divergent.

256. Under which of the following conditions is the sequence $\{x_n\}$ of real numbers convergent?
(1) The subsequences $\{x_{(2n+1)}\}$, $\{x_{2n}\}$ and $\{x_{3n}\}$ are convergent and have the same limit.
(2) The subsequences $\{x_{(2n+1)}$, $\{x_{2n}\}$ and $\{x_{3n}\}$ are convergent.
(3) The subsequences $\{x_{kn}\}_n$ are convergent for every k ≥ 2.
(4) $\lim_n |x_{(n+1)}-x_n|=0.$

257. Consider the following statements:
(a) Let f be a continuous function on [1, ∞) taking non-negative values such that $\int_1^\infty f(x)dx$ converges. Then $\sum_{n\geq1} f(n)$ converges.
(b) Let f be a function on [1, ∞) taking non-negative values such that $\int_1^\infty f(x)dx$ converges.
Then $\lim_{x\to\infty} f(x)=0.$
(c) Let f be a continuous, decreasing function on [1, ∞) taking non-negative values such that $\int_1^\infty f(x)dx$ does not converge. Then $\sum_{n\geq1} f(n)$ does not converge.
Which of the following options are true?
(1) (a), (b) and (c) are all true.
(2) Both (a) and (b) are false.
(3) (c) is true.
(4) (b) is true.

<u>Dec - 2023</u>

<u>PART – B</u>

258. Let f : ℝ → ℝ be a differentiable function such that f and its derivative f′ have no common zeros in [0, 1]. Which one of the following statements is true?
(1) f never vanishes in [0, 1].
(2) f has atmost finitely many zeros in [0, 1].
(3) f has infinitely many zeros in [0, 1].
(4) f(1/2) = 0.

259. Let f(x) be a cubic polynomial with real coefficients. Suppose that f(x) has exactly one real root and that this root is simple. Which one of the following statements holds for ALL antiderivatives F(x) of f(x)?
(1) F(x) has exactly one real root.
(2) F(x) has exactly four real roots.
(3) F(x) has atmost two real roots.
(4) F(x) has atmost one real root.

260. Consider the following infinite series:
(a) $\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{\sqrt{n}}$, (b) $\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^2}\right)$.
Which one of the following statements is true?

(1) (a) is convergent, but (b) is not convergent.
(2) (a) is not convergent, but (b) is convergent.
(3) Both (a) and (b) are convergent.
(4) Neither (a) nor (b) is convergent.

261. Let f : ℝ → ℝ be defined by

$$f(x) = \begin{cases} (1-x)^2 \sin(x^2), & x \in (0,1). \\ 0, & otherwise \end{cases}$$

and f′ be its derivative. Let

S = {c ∈ ℝ : f′(x) ≤ cf(x) for all x ∈ ℝ}.

Which one of the following is true?

(1) S = ∅
(2) S ≠ ∅ and S is a proper subset of (1, ∞)
(3) (2, ∞) is a proper subset of S
(4) S ∩ (0, 1) ≠ ∅

262. Consider the following subset of ℝ:

U = {x ∈ ℝ : $x^2 - 9x + 18 \le 0$, $x^2 - 7x + 12 \le 0$}.

Which one of the following statements is true?

(1) inf U = 5. (2) inf U = 4.
(3) inf U = 3. (4) inf U = 2.

263. Let X be a non-empty finite set and
Y = {$f^{-1}(0)$: f is a real-valued function on X}.
Which one of the following statements is true?
(1) Y is an infinite set
(2) Y has $2^{|x|}$ elements
(3) There is a bijective function from X to Y
(4) There is a surjective function from X to Y

264. Consider the sequence $(a_n)_{n\ge1}$,

where $a_n = \cos\left((-1)^n \frac{n\pi}{2} + \frac{n\pi}{3}\right)$.

Which one of the following statements is true?

(1) $\limsup_{n\to\infty} a_n = \frac{\sqrt{3}}{2}$.

(2) $\limsup_{n\to\infty} a_{2n} = 1$.

(3) $\limsup_{n\to\infty} a_{2n} = \frac{1}{2}$.

(4) $\limsup_{n\to\infty} a_{3n} = 0$.

PART – C

265. Suppose that f : [-1, 1] → ℝ is continuous. Which of the following imply that f is identically zero on [-1, 1]?

(1) $\int_{-1}^{1} f(x)x^n dx = 0$ for all n ≥ 0.

(2) $\int_{-1}^{1} f(x)p(x)dx = 0$ for all real polynomials p(x).

(3) $\int_{-1}^{1} f(x)x^n dx = 0$ for all n ≥ 0 odd.

(4) $\int_{-1}^{1} f(x)x^n dx = 0$ for all n ≥ 0 even.

266. Consider $\mathbb{R}^2$ with the Euclidean topology and consider $\mathbb{Q}^2 \subset \mathbb{R}^2$ with the subspace topology. Which of the following statements are true?

(1) $\mathbb{Q}^2$ is connected.
(2) If A is a non-empty connected subset of $\mathbb{Q}^2$, then A has exactly one element.
(3) $\mathbb{Q}^2$ is Hausdorff.
(4) $\{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}$ is compact in the subspace topology.

267. For a real number λ, consider the improper integrals

$$I_\lambda = \int_0^1 \frac{dx}{(1-x)^\lambda}, K_\lambda = \int_1^\infty \frac{dx}{x^\lambda}.$$

Which of the following statements are true?

(1) There exists λ such that I_λ converges, but K_λ does not converge.
(2) There exists λ such that K_λ converges, but I_λ does not converge.
(3) There exists λ such that I_λ, K_λ both converge.
(4) There exists λ such that neither I_λ nor K_λ converges.

268. Let f : ℝ → ℝ be a continuous function such that |f(x) – f(y)| ≥ log (1 + |x – y|) for all x, y ∈ ℝ. Which of the following statements are true?
(1) f is necessarily one-one.
(2) f need not be one-one.
(3) f is necessarily onto.
(4) f need not be onto.

269. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $p(x, y) = x$. Which of the following statements are true?

(1) Let $A_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Then for each $\gamma \in p(A_1)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_1)$.

(2) Let $A_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Then for each $\gamma \in p(A_2)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_2)$.

(3) Let $A_3 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$. Then for each $\gamma \in p(A_3)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_3)$.

(4) Let $A_4 = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$. Then for each $\gamma \in p(A_4)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_4)$

270. Let x be a real number. Which of the following statements are true?

(1) There exists an integer $n \geq 1$ such that $n^2 \sin\frac{1}{n} \geq x.$

(2) There exists an integer $n \geq 1$ such that $n \cos\frac{1}{n} \geq x.$

(3) There exists an integer $n \geq 1$ such that $ne^{-n} \geq x$.

(4) There exists an integer $n \geq 2$ such that $n(\log n)^{-1} \geq x$.

271. For real numbers a, b, c, d, e, f, consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (ax + by + c, dx + ey + f)$, for $x, y \in \mathbb{R}$.
Which of the following statements are true?

(1) F is continuous
(2) F is uniformly continuous
(3) F is differentiable
(4) F has partial derivatives of all orders

272. For a differentiable surjective function $f : (0, 1) \to (0, 1)$, consider the function $F : (0, 1) \times (0, 1) \to (0, 1) \times (0, 1)$ given by $F(x, y) = (f(x), f(y))$, $x, y \in (0, 1)$. If $f'(x) \neq 0$ for every $x \in (0, 1)$, then which of the following statements are true?

(1) F is injective.
(2) f is increasing.
(3) For every $(x', y') \in (0, 1) \times (0, 1)$, there exists a unique $(x, y) \in (0, 1) \times (0, 1)$ such that $F(x, y) = (x', y')$.
(4) The total derivative $DF(x, y)$ is invertible for all $(x, y) \in (0, 1) \times (0, 1)$.

273. Which of the following statements are true?

(1) The function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} [x]\sin\frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$
has a discontinuity at 0 which is removable.

(2) The function $f : [0, \infty) \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} \sin(\log x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$
has a discontinuity at 0 which is NOT removable.

(3) The function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} e^{1/x} & \text{for } x < 0, \\ e^{1/(x+1)} & \text{for } x \geq 0 \end{cases}$$
has a jump discontinuity at 0.

(4) Let $f, g : [0, 1] \to \mathbb{R}$ be two functions of bounded variation. Then the product fg has atmost countably many discontinuities.

274. Let $(f_n)_{n\geq 1}$ be the sequence of functions defined on [0, 1] by
$$f_n(x) = x^n \log\left(\frac{1+\sqrt{x}}{2}\right).$$
Which of the following statements are true?

(1) (f_n) converges pointwise on [0, 1].
(2) (f_n) converges uniformly on compact subsets of [0, 1) but not on [0, 1).
(3) (f_n) converges uniformly on [0, 1) but not on [0, 1].
(4) (f_n) converges uniformly on [0, 1].

275. Let $\{A_n\}_{n\geq 1}$ be a collection of non-empty subsets of $\mathbb{Z}$ such that $A_n \cap A_m = \varnothing$ for $m \neq n$. If $\mathbb{Z} = \cup_{n\geq 1} A_n$, then which of the following statements are necessarily true?

(1) A_n is finite for every integer $n \geq 1$.
(2) A_n is finite for some integer $n \geq 1$.
(3) A_n is infinite for some integer $n \geq 1$.
(4) A_n is countable (finite or infinite) for every integer $n \geq 1$.

276. Let $f : [0, \infty) \to \mathbb{R}$ be the periodic function of period 1 given by
$f(x) = 1 - |2x - 1|$ for $x \in [0, 1]$.

Further, define $g : [0, \infty) \to \mathbb{R}$ by $g(x) = f(x^2)$. Which of the following statements are true?

(1) f is continuous on $[0, \infty)$.
(2) f is uniformly continuous on $[0, \infty)$.
(3) g is continuous on $[0, \infty)$.
(4) g is uniformly continuous on $[0, \infty)$.

ANSWERS

1. (3)	2. (3)	3. (4)
4. (2)	5. (2)	6. (2)
7. (3)	8. (1,2)	9. (1,2,4)
10. (1,2,3)	11. (1,2)	12. (1,2)
13. (2,3,4)	14. (1)	15. (1,2,4)
16. (3)	17. (3)	18. (1)
19. (1)	20. (4)	21. (2,4)
22. (3,4)	23. (2,3)	24. (1,4)
25. (3,4)	26. (1,4)	27. (2,4)
28. (1)	29. (3)	30. (2)
31. (3)	32. (4)	33. (2)
34. (1,2,3,4)	35. (1,3)	36. (3,4)
37. (1,2)	38. (1,3)	39. (3,4)
40. (1,2)	41. (1,2,3)	42. (2,4)
43. (4)	44. (3,4)	45. (3,4)
46. (1)	47. (1)	48. (1)
49. (2)	50. (3)	51. (1)
52. (1,4)	53. (3,4)	54. (3,4)
55. (2,3)	56. (3,4)	57. (2,3,4)
58. (1,4)	59. (2,4)	60. (2)
61. (1)	62. (2)	63. (1)
64. (2)	65. (3)	66. (1,2)
67. (1)	68. (1,4)	69. (1,2,3)
70. (1,2,4)	71. (1,2,4)	72. (1,2)
73. (1,3,4)	74. (1,3)	75. (1)
76. (4)	77. (3)	78. (3)
79. (3)	80. (1)	81. (2,4)
82. (2,3)	83. (3)	84. (1,2,3)
85. (3,4)	86. (3,4)	87. (2)
88. (3)	89. (1,3,4)	90. (2)
91. (3)	92. (2)	93. (2)
94. (3)	95. (2)	96. (2)
97. (3)	98. (4)	99. (1,3,4)
100. (3,4)	101. (3,4)	102. (1,2,4)
103. (4)	104. (2,3,4)	105. (2)
106. (1,2,3)	107. (1,2,3)	108. (3,4)
109. (1,4)	110. (2)	111. (4)
112. (2)	113. (1)	114. (2)
115. (3)	116. (3)	117. (3)
118. (2)	119. (1)	120. (1,2,4)
121. (1)	122. (1,2)	123. (1,2,4)
124. (1,2)	125. (4)	126. (1,3,4)
127. (2)	128. (1,3)	129. (1,3,4)
130. (3)	131. (2)	132. (4)
133. (3)	134. (4)	135. (2)
136. (3)	137. (2,3)	138. (1,4)
139. (3,4)	140. (1,4)	141. (1,2,3,4)
142. (2,3)	143. (1,2,4)	144. (1,2,3)
145. (2,4)	146. (1,2,3)	147. (1,2)
148. (1,3,4)	149. (1,2,3)	150. (2)
151. (2)	152. (4)	153. (2)
154. (2)	155. (1)	156. (2,3)
157. (2,4)	158. (1,2,3,4)	159. (1,2)
160. (2,3)	161. (2,3)	162. (1,3)
163. (1,2,3)	164. (3,4)	165. (1,3)
166. (2)	167. (3)	168. (3)
169. (3)	170. (2)	171. (3)

172. (2)	173. (1,2,4)	174. (2,3,4)
175. (3)	176. (1,4)	177. (1,3,4)
178. (4)	179. (1)	180. (4)
181. (3,4)	182. (1,2,4)	183. (3,4)
184. (1,2,3,4)	185. (3)	186. (3)
187. (4)	188. (4)	189. (4)
190. (3)	191. (2,3)	192. (1,4)
193. (1,4)	194. (4)	195. (1,2,3,4)
196. (2)	197. (2,3,4)	198. (2,3,4)
199. (2)	200. (1,2,3)	201. (1,3)
202. (1)	203. (2)	204. (3)
205. (3)	206. (2)	207. (4)
208. (2)	209. (1,3)	210. (1,3)
211. (3,4)	212. (1,3)	213. (3,4)
214. (2,3)	215. (3)	216. (1,2)
217. (2,4)	218. (1)	219. (1)
220. (1,2)	221. (4)	222. (3)
223. (3)	224.	225. (4)
226.	227.	228.
229.	230.	231.
232.	233.	234.
235. (1,3,4)	236. (1,2)	237. (3)
238. (3,4)	239. (2)	240.(2)
241. (3)	242. (2,3)	243. (1)
244. (2)	245. (3)	246. (1)
247. (3,4)	248. (2)	249. (1,2,3)
250. (2)	251. (1,3)	252. (1,2,3,4)
253. (1,2,3,4)	254. (2,3,4)	255. (1,4)
256. (1,2)	257. (2,3)	258. (2)
259. (3)	260. (3)	261. (1)
262. (3)	263. (2)	264. (2)
265. (1,2)	266. (2,3)	267. (1,2,4)
268. (1,3)	269. (1,3,4)	270. (1,2,4)
271. (1,2,3,4)	272. (1,3,4)	273. (2,3,4)
274. (1,4)	275. (4)	276. (1,2,3)