NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 17, 2016

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions that follow

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 7 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Miscellaneous. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- **Calculators are not allowed.**

Notations

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

- The symbol S_n will stand for the set of all permutations of the symbols $\{1, 2, \cdots, n\}$, which is a group under composition.

- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.

- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval.

- We denote by $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$), the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).

- We denote by $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) the group (under matrix multiplication) of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and by $SL_n(\mathbb{R})$ (respectively, $SL_n(\mathbb{C})$), the subgroup of matrices with determinant equal to unity.

- The transpose (respectively, adjoint) of a matrix A in $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will be denoted by A^T (respectively, A^*). Similarly, the transpose of a (column) vector $b \in \mathbb{R}^n$ will be the (row) vector denoted by b^T.

- The derivative of a function f will be denoted by f' and the second derivative by f''.

Section 1: Algebra

1.1 Let $\alpha_i, i = 1, 2, 3$, be the roots of the polynomial

$$x^3 + px^2 + qx + r,$$

where $p, q, r \in \mathbb{C}$. Express the sum

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \alpha_i^2 \alpha_j$$

in terms of p, q and r.

1.2 How many subgroups of order 2 are there in S_4?

1.3 What is the maximal order of an element in S_7?

1.4 Let $G = GL_n(\mathbb{R})$. Which of the following subgroups are normal in G?
a. $H = SL_n(\mathbb{R})$.
b. $H =$ the set of all upper-triangular matrices in $GL_n(\mathbb{R})$.
c. $H =$ the set of all diagonal matrices in $GL_n(\mathbb{R})$.

1.5 In each of the cases where H is a normal subgroup, in Problem 1.4 above, identify the quotient group G/H.

1.6 Consider the ring $R = \mathcal{C}^1[-1, 1]$ of all continuously differentiable real valued functions on $[-1, 1]$ with the usual operations of pointwise addition and pointwise multiplication. In each of the cases below, state whether the given set is an ideal, a subring but not an ideal, or not a subring.
a. $\{f \in R \mid f(0) = 0\}$.
b. $\{f \in R \mid f'(0) = 0\}$.
c. $\{f \in R \mid f(0) = f'(0) = 0\}$.

1.7 List all integers n such that $1 \leq n \leq 10$ and such that there exists a field with n elements.

1.8 Let $W \subset \mathbb{M}_2(\mathbb{R})$ be the subspace of all matrices such that the entries of the first column add up to zero. Write down a basis for W.

1.9 Write down the inverse of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \end{bmatrix}.$$

1.10 Let $A \in \mathbb{M}_n(\mathbb{R})$ be a singular matrix. Let x_0 and b be vectors in $\mathbb{R}^n$ such that $Ax_0 = b$. Which of the following statements are true?
a. There exists $y_0 \in \mathbb{R}^n$ such that $A^T y_0 = b$.
b. There exist infinitely many solutions to the equation $Ax = b$.
c. If $A^T x = 0$, then it follows that $b^T x = 0$.

Section 2: Analysis

2.1 Let x_0 be an arbitrary positive real number. Define

$$x_{n+1} \;=\; \frac{1}{2}\left(x_n + \frac{5}{x_n}\right), n \geq 0.$$

Which of the following statements are true?
a. For all $n \geq 1$, we have $x_n \geq \sqrt{5}$.
b. The sequence $\{x_n\}_{n\geq 1}$ is monotonic.
c. The sequence $\{x_n\}_{n\geq 1}$ is convergent.

2.2 Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} z^n, z \in \mathbb{C}.$$

2.3 Let $f(x) = |x|^{\frac{5}{2}}, x \in \mathbb{R}$. Which of the following statements are true?
a. The function f is not differentiable at $x = 0$.
b. The function f is differentiable everywhere and $f'(x) = \frac{5}{2}|x|^{\frac{3}{2}}$.
c. The function f is differentiable everywhere and $f'(x) = \frac{5}{2}|x|^{\frac{1}{2}}x$.

2.4 Which of the following statements are true?
a. $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.
b. $|1 - \cos x| \leq |x|$ for all $x \in \mathbb{R}$.
c. $|\tan^{-1} x| \leq |x|$ for all $x \in \mathbb{R}$.

2.5 Which of the following functions are uniformly continuous over $\mathbb{R}$?
a. $f(x) = x^3$.
b. $f(x) = \int_0^x g(t)\, dt$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function.
c. $f(x) = \int_0^x g(t)\, dt$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function which is bounded.

2.6 Find the relationship between the height (h) and the radius (r) of the right circular cylinder of minimum total surface area amongst all right circular cylinders of volume 10cm^3.

2.7 Let $n \in \mathbb{N}$ and let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$.Which of the following statements are true?
a. J_n is rational for all $n \geq 2$.
b. J_n is irrational for all $n \geq 2$.
c. J_n is rational if $n = 7$.

2.8 Evaluate:

$$\int_0^5 \max\{x^2, 6x - 8\}\, dx.$$

2.9 Write down the power series expansion of the function $f(z) = \frac{1}{z^2}$ in a neighbourhood of $z = -1$.

2.10 Let $z = x + iy$. Which of the following functions are analytic in $\mathbb{C}$?
a. $f(z) = x - iy$.
b. $f(z) = e^x(\cos y - i \sin y)$.
c. $f(z) = e^{-x}(\cos y - i \sin y)$.

Section 3: Miscellaneous

3.1 Five persons, A,B,C,D and E, are to address a meeting.
a. If speaker A has to speak before speaker B, in how many ways can this be arranged?
b. In how many of these arrangements does speaker A speak immediately before speaker B?

3.2 Let $n \in \mathbb{N}$ be a fixed positive integer. Let $C_r = \binom{n}{r}, 0 \leq r \leq n$. Evaluate:

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \cdots + (-1)^n \frac{C_n}{n+1}.$$

3.3 Find a positive integer which is a multiple of 7 and which leaves a remainder 1 when divided by $2, 3, 4, 5$ and 6.

3.4 What is the remainder when 2^{1000} is divided by 13?

3.5 Let $2s$ be the perimeter of a triangle, where $s > 0$ is a fixed constant. What is the maximum possible value of the area of the triangle?

3.6 For $(x, y) \in \mathbb{R}^2$, let

$$T(x,y) = \left(\frac{x+y}{\sqrt{2}}, \frac{y-x}{\sqrt{2}} \right).$$

If D is the unit disc in the plane, what is the area of $T(D)$?

3.7 Let $S = \{(x,y) \in \mathbb{R}^2 \mid ax^2 + 2hxy + by^2 = 0\}$. In which of the following cases does S represent a pair of distinct straight lines?
a. $a = b = 5, h = -1$.
b. $a = b = 3, h = 4$.
c. $a = 1, b = 4, h = 2$.

3.8 Find the coordinates of the point in $\mathbb{R}^3$ which is the reflection of the point $(1, 2, 4)$ with respect to the plane $x + y + z = 1$.

3.9 A line segment moves in the plane with its end-points on the coordinate axes so that the sum of the cubes of the lengths of its intercepts on the coordinate axes is a constant c. Find the locus of the mid-point of this segment.

3.10 Which of the following statements are true?
a. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f(t) > 0$ for all $t > 0$, then f is continuous.
b. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Then $f(x) \geq 0$ for all $x \in \mathbb{R}$.
c. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $x = 0$ and such that $f(x) + f(x/2) = 0$ for all $x \in \mathbb{R}$. Then $f(x) = 0$ for all $x \in \mathbb{R}$.