NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 19, 2009

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 8 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the
 answer booklet, which is being supplied separately. This question
 paper is meant to be retained by you and so do not answer questions
 on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- N denotes the set of natural numbers, \mathbb{Z} the integers, \mathbb{Q} the rationals, \mathbb{R} the reals and \mathbb{C} the field of complex numbers. \mathbb{R}^n denotes the n-dimensional Euclidean space. The symbol]a,b[will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while [a,b] will stand for the corresponding closed interval; [a,b[and]a,b[will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. All logarithms, unless specified otherwise, are to the base e.
- Calculators are not allowed.

Section 1: Algebra

- 1.1 A polynomial in the variable x leaves a remainder 2 when divided by (x-3) and a remainder 3 when divided by (x-2). Find the remainder when it is divided by (x-2)(x-3).
- **1.2** Let $a_1, \dots, a_n \in \mathbb{R}$. Evaluate the determinant:

$$\begin{vmatrix} 1 + a_1 & a_2 & \cdots & a_n \\ a_1 & 1 + a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1 + a_n \end{vmatrix}.$$

1.3 Find the roots of the equation

$$27x^3 + 42x^2 - 28x - 8 = 0$$

given that they are in geometric progression.

1.4 Which of the following form a group?

a.

$$G = \left\{ \left(\begin{array}{cc} a & a \\ a & a \end{array} \right) : a \in \mathbb{R}, a \neq 0 \right\}$$

with respect to matrix multiplication.

b. \mathbb{Z}_4 , the set of all integers modulo 4, with respect to multiplication.

C.

$$G = \{f : [0,1] \to \mathbb{R} : f \text{continuous}\}$$

with respect to the operation defined by (f.g)(x) = f(x)g(x) for all $x \in [0,1]$.

- **1.5** Let G be a group and let H and K be subgroups of order 8 and 15 respectively. What is the order of the subgroup $H \cap K$?
- **1.6** Let G be a finite abelian group of odd order. Which of the following define an automorphism of G?
- a. The map $x \mapsto x^{-1}$ for all $x \in G$.
- b. The map $x \mapsto x^2$ for all $x \in G$.
- c. The map $x \mapsto x^{-2}$ for all $x \in G$.

1.7 An algebraic number is one which occurs as the root of a monic polynomial with rational coefficients. Which of the following numbers are algebraic?

a.
$$5 + \sqrt{3}$$

b.
$$7 + 2^{\frac{1}{3}}$$

c.
$$\cos \frac{2\pi}{n}$$
, where $n \in \mathbb{N}$

1.8 Given that the matrix

$$\begin{pmatrix} \alpha & 1 \\ 2 & 3 \end{pmatrix}$$

has 1 as an eigenvalue, compute its trace and its determinant.

- **1.9** Let A be a non-diagonal 2×2 matrix with complex entries such that $A = A^{-1}$. Write down its characteristic and minimal polynomials.
- 1.10 Pick out the true statements:
- a. Let A and B be two arbitrary $n \times n$ matrices. Then

$$(A+B)^2 = A^2 + 2AB + B^2.$$

b. There exist $n \times n$ matrices A and B such that

$$AB - BA = I.$$

c. Let A and B be two arbitrary $n \times n$ matrices. If B is invertible, then

$$tr(A) = tr(B^{-1}AB)$$

where tr(M) denotes the trace of an $n \times n$ matrix M.

Section 2: Analysis

2.1 Evaluate:

$$\lim_{x \to \infty} x \left(\log \left(1 + \frac{x}{2} \right) - \log \frac{x}{2} \right).$$

2.2 Evaluate:

$$\int_0^{\frac{\pi}{2}} \log \tan \theta \ d\theta.$$

2.3 Test the following series for convergence:

a.

$$\sum_{n=1}^{\infty} \left(n^{\frac{1}{n}} - 1 \right)^n$$

b.

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

where $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms.

2.4 Which of the following functions are continuous?

a.

$$f(x) = [x] + (x - [x])^{[x]}, x \ge \frac{1}{2},$$

where [x] denotes the largest integer less than, or equal to, x.

b.

$$f(x) = \lim_{n \to \infty} \frac{x^2 e^{nx} + x}{e^{nx} + 1}, \ x \in \mathbb{R}.$$

c.

$$f(x) = \lim_{n \to \infty} \frac{1}{n} \log (e^n + x^n), \ x \ge 0.$$

2.5 Write down the coefficient of x^7 in the Taylor series expansion of the function

$$f(x) = \log(x + \sqrt{1 + x^2})$$

about the origin.

2.6 Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable at x = a. Evaluate:

$$\lim_{n\to\infty} \left\{ f\left(a+\frac{1}{n^2}\right) + f\left(a+\frac{2}{n^2}\right) + \dots + f\left(a+\frac{n}{n^2}\right) - nf(a) \right\}.$$

2.7 Let

$$f(x,y) = x^4 - 2x^2y^2 + y^4 + x^2 - 6xy + 9y^2.$$

Examine whether f admits a local maximum or minimum at (0,0).

2.8 Let $z = x + iy \in \mathbb{C}$ and let f be defined by

$$f(z) = y - x - 3x^2i.$$

If C is the straightline segment joining z=0 to z=1+i, compute

$$\int_C f(z) \ dz.$$

2.9 Let C be the contour consisting of the lines $x=\pm 2$ and $y=\pm 2$, described counterclockwise in the plane. Compute

$$\int_C \frac{z}{2z+1} \ dz.$$

2.10 Let

$$f(z) = \frac{5z-2}{z(z-1)}.$$

Write down the residues of f at each of its poles.

Section 3: Geometry

- **3.1** Let M_1 ad M_2 be two points in the plane whose polar coordinates are given as $(12, 4\pi/9)$ and $(12, -2\pi/9)$ respectively. Find the polar coordinates of the midpoint of the line segment joining these points.
- **3.2** The length of the sides of a rhombus is given as $5\sqrt{2}$. If two of its opposite vertices have coordinates (3, -4) and (1, 2), find the length of the altitude of the rhombus.
- **3.3** What figure does the equation

$$\sum_{i,j=1}^{2} a_{ij} x_i x_j = 1$$

represent when $A = (a_{ij})$ is the matrix given by

a.

$$A = \left(\begin{array}{cc} 5 & 3 \\ 3 & 2 \end{array}\right)?$$

b.

$$A = \left(\begin{array}{cc} 2 & 3 \\ 3 & 2 \end{array}\right)?$$

- **3.4** Find the locus of a point which moves such that the ratio of its distance from the point (-5,0) to its distance from the line 5x + 16 = 0 is 5/4.
- **3.5** Express the equations of the curves given below in parametric form in the form f(x,y) = 0.

a.

$$x = \frac{a}{2} \left(t + \frac{1}{t} \right), \ y = \frac{b}{2} \left(t - \frac{1}{t} \right).$$

b.

$$x = 2R\cos^2 t, \ y = R\sin 2t.$$

3.6 Let A_n be the area of the polygon whose vertices are given by the n-th roots of unity in the complex plane. Evaluate:

$$\lim_{n\to\infty} A_n.$$

3.7 Write down the equation of the diameter of the sphere

$$x^2 + y^2 + z^2 + 2x - 6y + z - 11 = 0$$

which is perpendicular to the plane 5x - y + 2z = 17.

- **3.8** Find the values of a for which the plane x + y + z = a is tangent to the sphere $x^2 + y^2 + z^2 = 12$.
- **3.9** Let d(P,Q) denote the distance between two points P and Q in the plane. Let

$$A = \{(x,y) \in \mathbb{R}^2 : xy = 0\}, \text{ and } B = \{(x,y) \in \mathbb{R}^2 : xy = 1\}.$$

Compute:

$$\inf_{p \in A, Q \in B} d(P, Q).$$

3.10 A ray, having the origin as its end-point, initially coincides with the x-axis and rotates about the origin in the plane with constant angular velocity ω . A point starts at the origin and moves along the ray with constant velocity v. Write down the parametric equations of the locus of the point in the form $x = \varphi(t), y = \psi(t)$, with time t as the parameter.