NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 19, 2015

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 8 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Miscellaneous. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- **Notations**
  - $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
  - The symbol $\mathbb{Z}_n$ will denote the ring of integers modulo n.
  - The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.
  - The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
  - We denote by $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$), the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).
  - The trace of a square matrix A will be denoted tr(A) and the determinant by det(A).
  - The derivative of a function f will be denoted by f' and the second derivative by f''.
  - All logarithms, unless specified otherwise, are to the base e.
- **Calculators are not allowed.**

Section 1: Algebra

1.1 Which of the following statements are true?
a. If G is a group such that $(ab)^2 = a^2b^2$ for all $a, b \in G$, then G is abelian.
b. If G is a group such that $a^2 = e$ for all $a \in G$, where e is the identity element in G, then G is abelian.
c. If G is a group such that $a^2 = e$ for all $a \in G$, where e is the identity element in G, then G is finite.

1.2 For real numbers a and b, define the mapping $\tau_{a,b} : \mathbb{R} \to \mathbb{R}$ by $\tau_{a,b}(x) = ax + b$. Let

$$G = \{\tau_{a,b} \ : \ a, b \in \mathbb{R}, a \neq 0\}.$$

Under composition of mappings, this becomes a group. Which of the following subgroups of G are normal?
a. $H = \{\tau_{a,b} \mid a \neq 0, a \in \mathbb{Q}, b \in \mathbb{R}\}$.
b. $H = \{\tau_{1,b} \mid b \in \mathbb{R}\}$.
c. $H = \{\tau_{1,b} \mid b \in \mathbb{Q}\}$.

1.3 Find all positive integers $n > 1$ such that the polynomial $x^4 + 3x^3 + x^2 + 6x + 10$ belongs to the ideal generated by the polynomial $x^2 + x + 1$ in $\mathbb{Z}_n[x]$.

1.4 Find the number of irreducible monic polynomials of degree 2 over the field $\mathbb{F}_5$ of five elements.

1.5 Find the number of invertible 2×2 matrices with entries in $\mathbb{F}_2$, the field of two elements.

1.6 Let V be a finite dimensional vector space and let A, B and C be subspaces of V. Which of the following statements are true?
a. $A \cap (B + C) = A \cap B + A \cap C$.
b. $A \cap (B + C) \subset A \cap B + A \cap C$.
c. $A \cap (B + C) \supset A \cap B + A \cap C$.

1.7 Write down the inverse of the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

1.8 Find an orthogonal matrix P such that $PAP^{-1} = B$, where

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

1.9 Let $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the value of $\det(A)$ when (i) $n = 10$ and (ii) $n = 100$?

1.10 Let $n > 1$ be a positive integer and let $A \in \mathbb{M}_n(\mathbb{R})$ be as defined in Problem 1.9 above. Write down the set of eigenvalues of A.

Section 2: Analysis

2.1 In each of the following cases, state whether the given series is absolutely convergent, conditionally convergent or divergent.

a.

$$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots$$

b.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}.$$

c.

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)}.$$

2.2 Evaluate:

$$\lim_{\theta \to 0} (1 - 2\tan\theta)^{\cot\theta}.$$

2.3 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $t \in \mathbb{R}$. Evaluate:

$$\lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t+h} f(s)\ ds.$$

2.4 Evaluate:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}.$$

2.5 Define:

$$f(x) \;=\; \begin{cases} a_n + \sin \pi x & \text{if } x \in [2n, 2n+1], \\ b_n + \cos \pi x & \text{if } x \in]2n-1, 2n[, \end{cases}$$

where n varies over $\mathbb{N} \cup \{0\}$. Find all possible sequences $\{a_n\}$ and $\{b_n\}$ such that f is continuous on $[0, \infty[$.

2.6 Which of the following functions are uniformly continuous on the prescribed domain?
a. $f(x) = x \sin \frac{1}{x}$ on $]0, 1[$.
b. $f(x) = e^x \cos \frac{1}{x}$ on $]0, 1[$.
c. $f(x) = \sin(\sin x)$ on $]0, \infty[$.

2.7 Let k be a positive integer. Let $a > 0$. Evaluate:

$$\lim_{n \to \infty} \frac{(a + \frac{1}{n})^n (a + \frac{2}{n})^n \cdots (a + \frac{k}{n})^n}{a^{nk}}.$$

2.8 Which of the following functions defined on $\mathbb{R}$ are differentiable?
a. $f(x) = x|x|$.
b. $f(x) = [x] \sin^2 \pi x$.
c. $f(x) = \sqrt{|x|}$.
(The symbol $[x]$ in statement (b) stands for the integral part of $x \in \mathbb{R}$, *i.e.* the largest integer less than, or equal to, x.)

2.9 Which of the following statements are true?
a. Let f be continuously differentiable in $[a, b]$ and twice differentiable on $]a, b[$. If $f(a) = f(b)$ and if $f'(a) = 0$, then there exists $x_0 \in]a, b[$ such that $f''(x_0) = 0$.
b. Let f be continuously differentiable in $[a, b]$. If $f(a) = f(b)$ and if $f'(a) = f'(b)$, then there exists x_1 and x_2 in $]a, b[$ such that $x_1 \neq x_2$ and such that $f'(x_1) = f'(x_2)$.
c. Let f be continuously differentiable on $[0, 2]$, and twice differentiable on $]0, 2[$. If $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then there exists $x_0 \in]0, 2[$ such that $f''(x_0) = 0$.

2.10 Let Γ be the boundary of the square in the complex plane with vertices at the points $0, 1, 1+i$ and i, which is described in the anticlockwise direction. Evaluate:

$$\int_\Gamma \pi e^{\pi \bar{z}} \, dz.$$

Section 3: Miscellaneous

3.1 Which of the following relations are true?
a. $(-1)^{\frac{n(n-1)}{2}} = (-1)^{\frac{n(n+1)}{2}}$.
b. $(-1)^{\frac{n(n-1)}{2}} = (-1)^{[\frac{n}{2}]}$.
c. $(-1)^{\frac{n(n-1)}{2}} = (-1)^{n^2}$.
(The symbol $[\frac{n}{2}]$ in statement (b) stands for the integral part of $\frac{n}{2}$; see also Problem 2.8.)

3.2 Let n, k and r be positive integers such that $k < r < n$ and also such that $n > r + k$. Which of the following statements are true?
a.
$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}.$$
b.
$$\binom{n}{r} = \sum_{l=0}^{k}\binom{k}{l}\binom{n-k}{r-l}.$$
c.
$$\binom{n}{r} = \sum_{l=0}^{k}\binom{k}{l}\binom{n-l}{r-k}.$$

3.3 Let $a, b, c \in \mathbb{R}$. Evaluate the determinant:
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}.$$

3.4 Find the sum of the following series up to n terms:
$$\frac{1}{2.3}2 + \frac{2}{3.4}2^2 + \frac{3}{4.5}2^3 + \cdots$$

3.5 Let $N = 7776$. Find the number of divisors of N, excluding 1 and N.

3.6 Let P_n be the polygon in the complex plane with vertices situated at the n-th roots of unity. Let A_n be its area and let L_n be its perimeter. Evaluate:
$$\lim_{n\to\infty}\frac{L_n^2}{A_n}.$$

3.7 Let $a, b \in \mathbb{R}$. For $P = (x, y) \in \mathbb{R}^2$, define $f(P) = ax + by$. The line $ax + by = 4$ intersects the line segment joining two points P_1 and P_2 in the plane at a point R such that $P_1R : RP_2 = 1 : 2$. If $f(P_1) = 3$, what is the value of $f(P_2)$?

3.8 Let $n \in \mathbb{N}$. Which of the following statements are true?
a. $x^n - 1 \geq n(x-1)x^{\frac{n-1}{2}}$ for all $x \geq 0$.
b. $x^n - 1 \geq n(x-1)x^{\frac{n-1}{2}}$ for all $x \geq 1$.
c. $x^n - 1 \geq n(x-1)x^{\frac{n-1}{2}}$ for all $x \geq n$.

3.9 What is the radius of the sphere with centre at the origin and which has the plane $x + y + z = 1$ as a tangent?

3.10 Let

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 2hxy + y^2 = 1\}.$$

For what values of h is the set S non-empty and bounded?